Bounded Transport in Tilings

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The question

Given a (repetitive, uniquely ergodic, aperiodic) tiling T and two locally-determined mass distributions f_1, f_2 on T:

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- When can it be done in a weakly pattern-equivariant way?
- What does this have to do with cohomology?

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- The distribution f is *weakly pattern equivariant* (wPE) if it is the uniform limit of sPE distributions.
- A *bounded transport* from f_1 to f_2 is a set of rules for moving mass around (e.g. send 10 kg from tile t_1 to tile t_2, 3kg from t_2 to t_3, and 5 kg from t_3 to t_1) such that
  - The motion changes f_1 to f_2, and
  - No mass is moved by more than a fixed distance D

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- wPE is the uniform limit of sPE. For any ϵ there exists R_ϵ such that the transport from t_i to t_j is determined within ϵ by the R_ϵ neighborhood of t_i.

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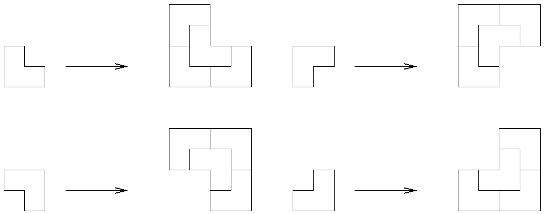
Musical chairs



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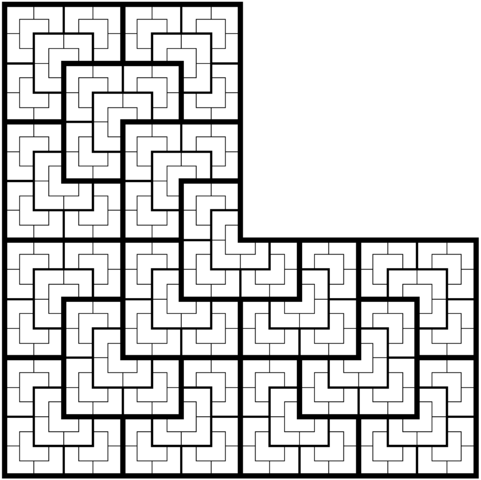

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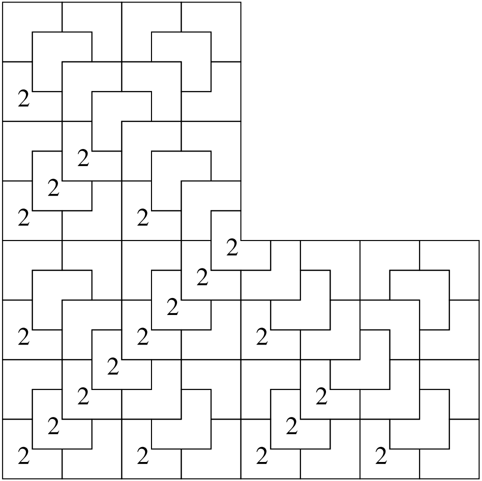
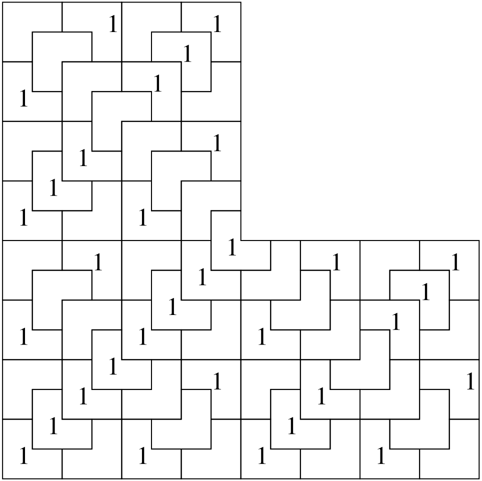
- All three distributions have overall density 0.5 kg/tile. Which are related by bounded/wPE/sPE transport?

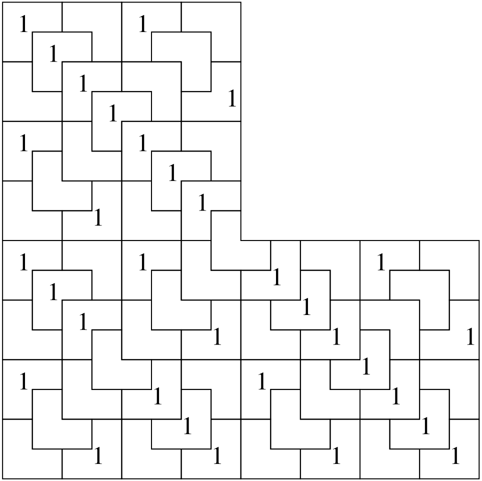
2 kg on the NE chairs



1 kg on the NE and SW chairs



1 kg on the NW and SE chairs



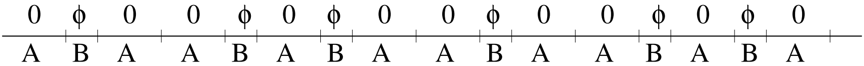
A Fibonacci mass distribution

Put 1kg on each A tile:

1	0	1	1	0	1	0	1	1	0	1	1	0	1	0	1
A	B	A	A	B	A	B	A	A	B	A	A	B	A	B	A

Another Fibonacci mass distribution

Put $\phi = (1+\sqrt{5})/2$ kg on each B tile.



Same overall density. Is there bounded/wPE/sPE transport?

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Pattern-equivariant cochains

- A tiling T gives a decomposition of $\mathbb{R}^n$ into vertices, edges, 2-cells, 3-cells, etc. Tiles are n-cells. Orient the cells arbitrarily.
- A (real-valued) k-cochain assigns a real number to each oriented k-cell. A mass distribution is just an n-cochain.
- k-cochains can be bounded/wPE/sPE.
- Coboundaries: If α is a k-cochain, and c is a $(k+1)$-cell, then $(\delta\alpha)(c) := \alpha(\partial c)$.
- If α is bounded/wPE/sPE, so is $\delta\alpha$, albeit with bigger radius.
- Let Ω^k_w and Ω^k_s denote the weakly and strongly PE k-cochains on T.

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- $H^k_{PE} = \dfrac{\text{Closed } k\text{-cochains}}{\text{Exact } k\text{-cochains}}$
- A cohomology class is *asymptotically negligible (AN)* if it can be represented by a weakly exact cochain.

A topological invariant

Theorem (Kellendonk-Putnam, S)

If T is a repetitive tiling, then H^k_{PE} is canonically isomorphic to the k-th real-valued Čech cohomology $\check{H}^k(\Omega_T)$, where Ω_T is the continuous hull of T. In particular, all tilings in Ω_T have the same PE cohomology.

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A lot is known about cohomology of tiling spaces. If we can reduce questions of transport to questions of cohomology, we win.

Integration and well-balanced cochains

- k-cochains are made to be integrated on k-chains. If α is an n-cochain and X is a union of tiles, then

$$\alpha(X) = \sum_{t \in X} \alpha(t).$$

- α is *well-balanced* if $\exists C < \infty$ such that, for any finite union X of tiles,

$$|\alpha(X)| \leq C|\partial X|.$$

- Adding $\delta\beta$ to α does not change well-balancedness, since $|\delta\beta(X)| = |\beta(\partial X)| \leq C|\partial X|$. Well-balancedness is a property of cohomology *classes*, not just cochains.

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Cohomological answers to transport questions

If f_1 and f_2 are mass distributions on T, then f_1 and f_2 are closed and define cohomology classes $[f_1]$ and $[f_2]$. Then

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- There is a wPE transport from f_1 to f_2 if and only if $f_1 - f_2$ is weakly exact.
- There is a sPE transport from f_1 to f_2 if and only if $f_1 - f_2$ is exact, i.e. if and only if $[f_1] = [f_2]$.

Laczkovich and Hall

- For fixed D, let X_{+D} be the union of all tiles within distance D of the tiles of X.
- If there is a bounded transport from f_1 to f_2 moving mass a maximum distance D, then for any X, $f_1(X_{+D}) \geq f_2(X)$ and $f_2(X_{+D}) \geq f_1(X)$.
- Using the Hall Marriage Theorem, Laczkovich proved that this condition is sufficient for the existence of bounded transport.
- We need to show that the Laczkovich condition is equivalent to well-boundedness.

Trains and border crossings

Imagine supply depots in each tile. WLOG assume that f_1 and f_2 are strictly positive and bounded below by ϵ. Ship goods by train to nearby tiles. Customs agents keep track of how much mass crosses each border.

- If $f_1 - f_2$ is well-balanced, and if $C < N\epsilon$ for some integer N, and if the tiles have a maximum of M faces, then we can accomplish the mass transport from f_1 to f_2 by interpolating linearly in NM steps. At each step the Lackovich condition is met with D being the diameter of the largest tile.
- If f_1 and f_2 satisfy the Laczkovich condition, then there is a bounded transport. For each edge e let $\beta(e)$ be the net amount of mass crossing the edge. Then $\delta\beta = f_1 - f_2$. Since β is bounded, $|(f_1 - f_2)(X)| = |\beta(\partial X)| \leq C|\partial X|$.

PE border crossings

- If the transport is (weakly or strongly) PE, then β is (weakly or strongly) PE, and $f_1 - f_2$ is (weakly or strongly) exact.
- Conversely, if $f_1 - f_2 = \delta\beta$ for some PE cochain β, then β gives the instructions for mass transport. If there's not enough mass in some tiles to transport across an edge in one step, use multiple steps.

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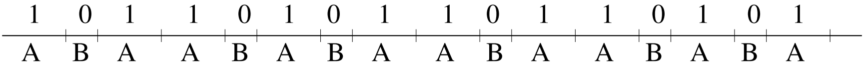
Fibonacci puzzle



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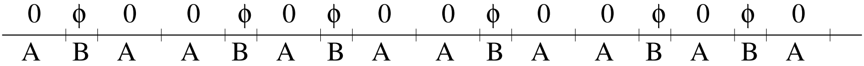

Fibonacci answer

- For the Fibonacci tiling, $\check{H}^1(\Omega_T) = \mathbb{Z}^2$, so $\check{H}^1(\Omega_T, \mathbb{R}) = \mathbb{R}^2$.
- $H^1_{AN} = \mathbb{R}$. The difference of two sPE mass distributions is weakly exact if and only if they have the same density.
- So there is a wPE transport, and hence a bounded transport. But is there a sPE transport?

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- So there is a wPE transport, and hence a bounded transport. But is there a sPE transport?
- No! If $f_1 - f_2 = \delta\beta$ and β is sPE, then β takes on the exact same value repeatedly, say at positions a and b. But then $(f_1 - f_2)[a, b] = 0$. But $(f_1 - f_2)$ is a positive multiple of 1 minus a positive multiple of ϕ, and cannot be zero.

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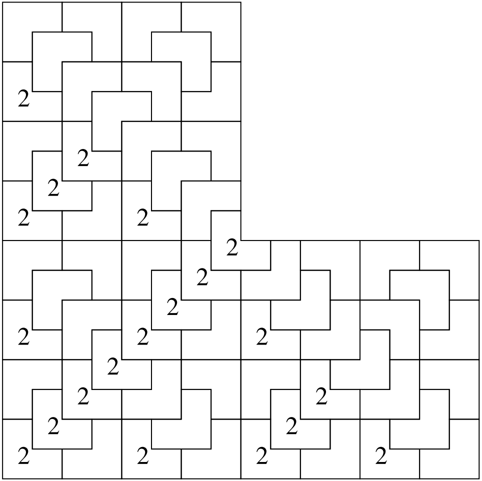

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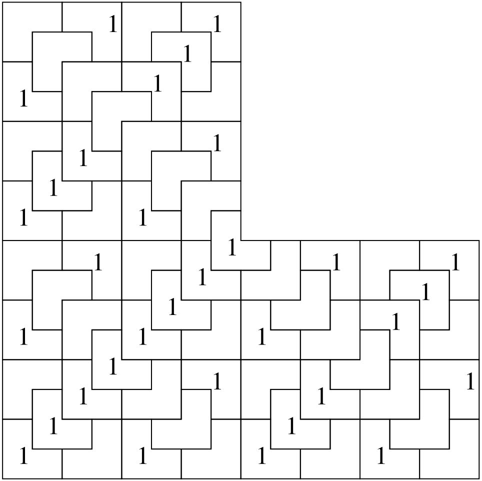

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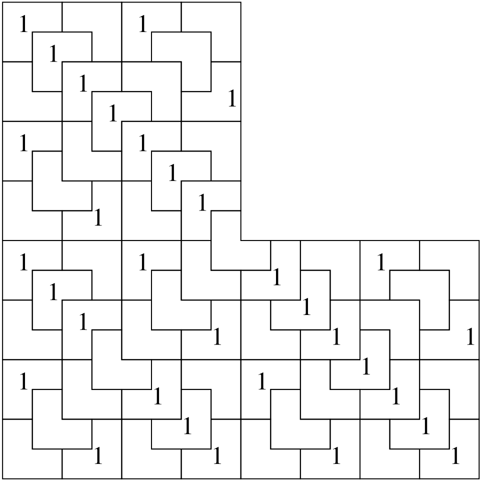

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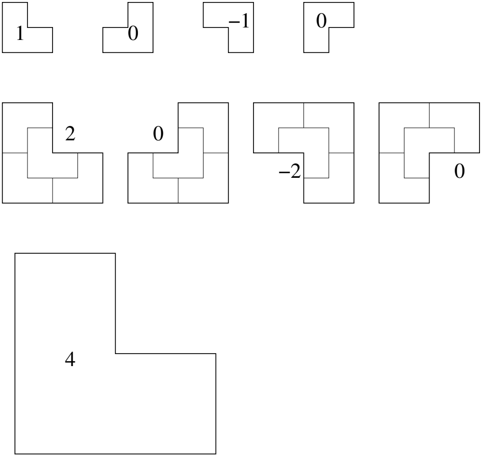
- One generator counts NE minus SW. This is $f_1 - f_2$. Not weakly exact, so there is no wPE transport.
- (Last generator counts NW minus SE.)
- Remaining question: Is $f_1 - f_2$ well-balanced?

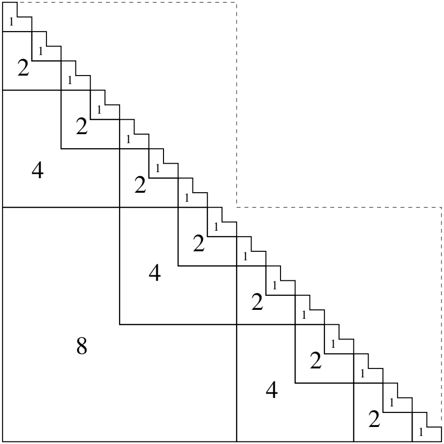
Scaling properties

Under substitution, $f_1 - f_2$ doubles at each stage:



N log N

On triangle of side length $N = 2^m$, $f_1 - f_2$ goes as $m2^m$.



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- But what about the converses?

sPE vs. wPE

Existence of wPE transport does not imply existence of sPE transport.

AN classes exist. Fibonacci puzzle has a wPE solution but not a sPE solution.

wPE vs. bounded transport

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Take the standard square lattice on $\mathbb{R}^2$. Let $f_1 = f_2 =$ constant. (E.g. 1kg on each tile). Move mass from $(0,0)$ to $(0,1)$ to $(1,1)$ to $(1,0)$ to $(0,0)$.

- Initial and final mass distributions are sPE. (Constant!)
- Transport process is not wPE.

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- Initial and final mass distributions are sPE. (Constant!)
- Transport process is not wPE.
- Problem is gauge freedom. Need a rule to fix the curl around each vertex.

Conjecture 1

Let f_1 and f_2 be sPE mass distributions for a repetitive and uniquely ergodic tiling T. If there exists a bounded transport from f_1 to f_2, then there exists a (possibly different) wPE transport from f_1 to f_2.

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Cohomological restatement:

If α is a well-balanced sPE n-cochain on Ω_T, then α is asymptotically negligible.

Expanding the question

Generalize definitions of well-bounded and asymptotically negligible to cohomology classes of all degrees:

- A closed k-cochain is *well-balanced if its integral over any k-chain X (union of k-faces of tiles in T) is bounded by the boundary of X.*
- *A class in H^k is well-balanced if it is represented by a well-balanced cochain.*
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- *A class is H^k is asymptotically negligible if it is represented by a weakly exact cochain.*

More general conjecture: Every well-balanced cohomology class of arbitrary degree is asymptotically negligible.

Status of the conjecture

- True when $k = 1$, independent of n. (Gottschalk-Hedlund theorem. Tiling version by Kellendonk-S) A closed sPE 1-cochain is weakly exact if and only if its integral is bounded.

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- Partial results when T is a self-similar (substitution) tiling.

Results for substitution tilings

- T is a substitution tiling with stretching factor λ. Substitution map $\sigma : \Omega_T \to \Omega_T$ induces map $\sigma^* : H^k(\Omega_T, \mathbb{R}) \to H^k(\Omega_T, \mathbb{R})$. Decompose $H^k(\Omega_T, \mathbb{R})$ into (generalized) eigenspaces of σ^*.

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Open question

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Answer in time for Jean-Marc's 70th birthday.

Merci de votre attention!