Frenkel-Kontorova model in almost-periodic environments

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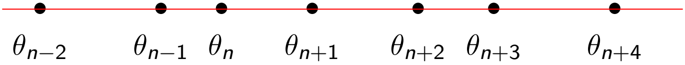
Frenkel-Kontorova Model, 1938



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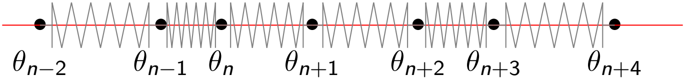

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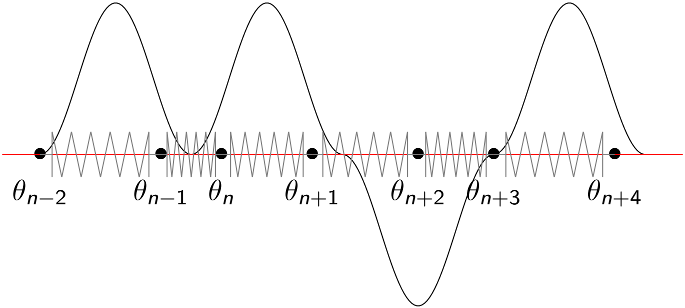

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1. $E : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is C^2.
2. *Twist*: for some $\alpha > 0$, $\frac{\partial^2 E}{\partial x \partial y} \leq -\alpha < 0$.
3. *translation bounded*: $\forall R > 0$, $\sup_{|x-y|<R} E(x, y) < +\infty$.

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Original ex:

$$E(x,y) = W(x-y) + V(x),$$

$$W(x-y) = \frac{1}{2}|x-y-\lambda|^2 \text{ and } V(x) = \frac{K}{2\pi}(1-\cos(2\pi x))$$

for some constants K, λ.

Frenkel-Kontorova Model, 1938

A configuration $(\theta_n)_{n\in\mathbb{Z}}$ is said minimizing if for any segment $(\theta_m, \theta_{m+1}, \dots, \theta_n)$

$$\sum_{j=m}^{n-1} E(\theta_i, \theta_{i+1}) \leq \sum_{j=m}^{n-1} E(\theta'_i, \theta'_{i+1})$$

for any segment $\theta'_m < \theta'_{m+1} < \cdots < \theta'_n$ with $\theta'_m = \theta_m$ and $\theta'_n = \theta_n$.

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Q. What are the properties of minimizing configurations? (Existence ?,...)

A minimizing configuration is critical.

Euler-Lagrange Equation

$$\frac{\partial E}{\partial y}(\theta_{k-1}, \theta_k) + \frac{\partial E}{\partial x}(\theta_k, \theta_{k+1}) = 0.$$

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$$\frac{\partial E}{\partial x}(\theta_k, \cdot)^{-1}\left(-\frac{\partial E}{\partial y}(\theta_{k-1}, \theta_k)\right) = \theta_{k+1}.$$

Euler-Lagrange map

$$\begin{pmatrix} \theta_k \\ \theta_{k-1} \end{pmatrix} \mapsto \begin{pmatrix} \theta_{k+1} \\ \theta_k \end{pmatrix} = \begin{pmatrix} \frac{\partial E}{\partial x}(\theta_k, \cdot)^{-1}(-\frac{\partial E}{\partial y}(\theta_{k-1}, \theta_k)) \\ \theta_k \end{pmatrix}$$

Theorem (Aubry-Le Daeron; Mather, 1983)

For the energy $E(x, y) = W(x - y) + V(x)$ *with* V *1-periodic:* $V(\cdot + 1) = V(\cdot)$

i) *Any minimizing configuration* $(\theta_n)_n$ *has a rotation number* $\rho((\theta_n)_n)$,

$$\lim_{n \to \pm\infty} \frac{\theta_n}{n} = \rho.$$

ii) *The rotation number* $\rho((\theta_n)_n)$ *depends continuously on* $(\theta_n)_n$ *(for the product topology).*

iii) *Any real* $\rho \geq 0$ *is the rotation number of some minimizing configuration.*

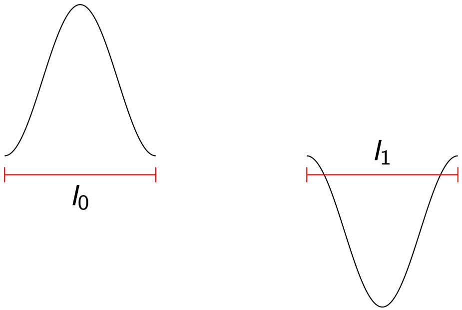
An energy $E(x, y) = W(x - y) + V(x)$ is an almost crystalline interaction if
Setting for any interval $I \subset \mathbb{R}$

$$\mathcal{R}_I := \{x \in \mathbb{R} : V(\cdot)|_I = V(\cdot + x)|_I\},$$

- *Finite complexity*: There are finitely many intervals $I_1, \dots, I_n$ s.t. $\mathcal{R}_{I_i}$ is discrete and $\mathbb{R} = \bigcup_{i=1}^{n} \mathcal{R}_{I_i} + I_i$.
- *Repetitivity*: Every set $\mathcal{R}_I$ is relatively dense.
- *Uniform density*: Every $\mathcal{R}_{I_i}$ admits a density: the limit $\lim_{n \to \pm\infty} \frac{\#\mathcal{R}_I \cap [-n, n]}{2n} > 0$ exists.

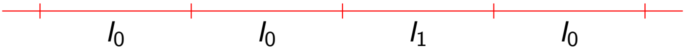
Almost crystalline interaction: example

Let V be defined on two intervals I_0, I_1



Almost crystalline interaction: example

A sequence $(s_n)_n \in \{0;1\}^{\mathbb{Z}}$ codes a tiling T of $\mathbb{R}$ by intervals I_0, I_1.



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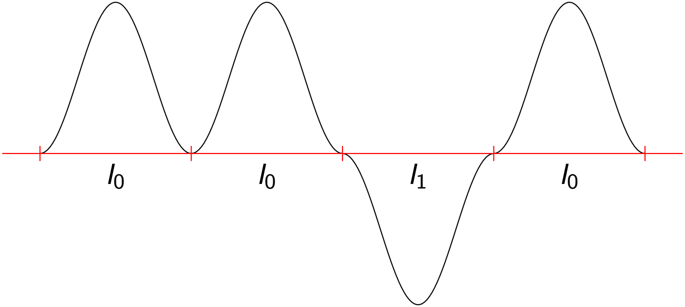

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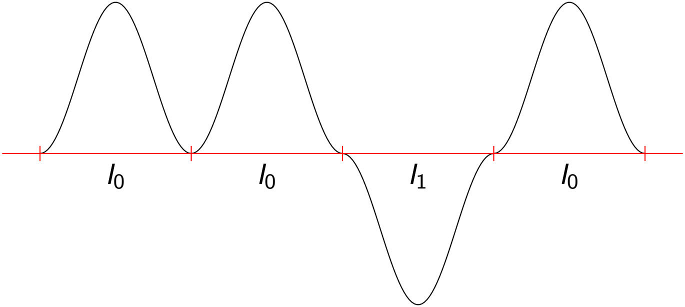

When $s_n = \lfloor n\alpha \rfloor - \lfloor (n-1)\alpha \rfloor$, $\alpha \in (0,1)$,
this defines an almost crystalline interaction.

Theorem (Gambaudo-Guiraud-P.,06)

For an almost crystalline interaction $E(x, y) = W(x - y) + V(x)$.

i) *Any minimizing configuration* $(\theta_n)_n$ *has a rotation number* $\rho((\theta_n)_n)$,

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The ground energy is

$$\bar{E} := \lim_{n\to+\infty} \inf_{x_0,\dots,x_n\in\mathbb{R}} \frac{1}{n}\sum_{i=0}^{n-1} E(x_i, x_{i+1}).$$

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Existence by subadditivity.

A configuration $(\theta_n)_n$ is said calibrated (at the level $\bar{E}$) if for any $m \leq n$

$$\sum_{i=m}^{n-1} \left[E(\theta_i, \theta_{i+1}) - \bar{E} \right] \leq \inf_{\ell \geq 1} \inf_{y_0 = \theta_m, \ldots, y_\ell = \theta_n} \sum_{i=0}^{\ell-1} \left[E(y_i, y_{i+1}) - \bar{E} \right].$$

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A calibrated configuration is a minimizing configuration.

Aubry-Mather theory

Theorem (Aubry, Mather, 83, 89, ...)

For the energy $E(x, y) = W(x - y) + V(x)$ *with* V *1-periodic.*

- *minimizing configurations are calibrated configuration for some* $E_\lambda(x, y) = E(x, y) - \lambda(y - x)$, $\lambda \in \mathbb{R}$

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$$\rho = \lim_{n \to \pm\infty} \frac{\theta_n}{n} = -\frac{d\bar{E}_\lambda}{d\lambda}.$$

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$$\rho = \lim_{n \to \pm\infty} \frac{\theta_n}{n} = -\frac{d\bar{E}_\lambda}{d\lambda}.$$

- *If* $\rho \notin \mathbb{Q}$, *then* $\Lambda(\rho) := \{\lambda : \rho = -\frac{d\bar{E}_\lambda}{d\lambda}\}$ *has empty interior.* *If* $\rho \in \mathbb{Q}$, *then* $\mathrm{int}(\Lambda(\rho)) \neq \emptyset$

$$\sup_{n,k \in \mathbb{Z}} |\theta_{n+k} - \theta_k - n\rho| < +\infty.$$

Aubry-Mather theory

Theorem (Garibaldi-P.-Thieullen, 2017)

For an almost crystalline interaction $E(x, y) = W(x - y) + V(x)$*. There exist configurations* $(\theta_n)_{n\in\mathbb{Z}}$ *calibrated for* E *with bounded jumps:*

$$\sup_{n\in\mathbb{Z}} |\theta_{n+1} - \theta_n| < +\infty.$$

For the energy $E(x,y) = W(x,y) + V(x)$ with V 1-periodic.
Effective potential:[Chou-Griffiths, 86] A C^0 periodic function u solution of the problem

$$\begin{cases} u(y) + \bar{E} = \min_x [u(x) + E(x,y)], & \forall y \text{ (backward)} \\ u(x) + \bar{E} = \max_y [u(y) - E(x,y)], & \forall x \text{ (forward)} \end{cases}$$

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Remark: u plays the role of a discrete *viscosity solution* for the Hamilton-Jacobi equation (sometimes also called calibrated solution or corrector).

weak KAM theory: Periodic case

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Remark: u plays the role of a discrete *viscosity solution* for the Hamilton-Jacobi equation (sometimes also called calibrated solution or corrector).

Existence by a fixed point theorem for a Lax-Oleinik operator.

weak KAM theory: Periodic case

A configuration $(\theta_n)_n$ such that for some effective potential

$$u(\theta_{n+1}) + \bar{E} = u(\theta_n) + E(\theta_n, \theta_{n+1}), \qquad \forall n \in \mathbb{Z}$$

is calibrated:

$$\sum_{i=m}^{n-1} E(\theta_i, \theta_{i+1}) - \bar{E} = u(\theta_n) - u(\theta_m) \leq \sum_{i=m}^{n-1} E(y_i, y_{i+1}) - \bar{E}$$

for all configuration $(y_k)_k$.

weak KAM theory: almost crystalline case

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Mañé potential

$$S(x,y) = \inf_{n\geq 1} \inf_{x_0=x,\ldots,x_n=y} \sum_{i=0}^{n-1} E(x_{i+1},x_i) - \bar{E}.$$

S is a semi-distance $S(x,y) \leq S(x,z) + S(z,y)$

$$S(x_m,x_n) \leq \sum_{i=m}^{n-1} E(x_{i+1},x_i) - \bar{E}. \tag{1}$$

Existence of configuration $(x_k)_k$ with equality in (1) by using *minimizing measure*.

weak KAM theory: almost crystalline case

Theorem (Garibaldi-P.-Thieullen, 2018+)

For a good almost crystalline interaction
$E(x,y) = W(x-y) + V(x)$.
There exists a bounded effective potential $u\colon \mathbb{R} \to \mathbb{R}$

$$\begin{cases} u(y) + \bar{E} = \min_x [u(x) + E(x,y)], & \forall y \ \textit{(backward)} \\ u(x) + \bar{E} = \max_y [u(y) - E(x,y)], & \forall x \ \textit{(forward)} \end{cases}$$

An energy $E(x, y) = W(x - y) + V(x)$ is an good almost crystalline interaction if for any interval $I \subset \mathbb{R}$

$$\mathcal{R}_I := \{x \in \mathbb{R} : V(\cdot)|_I = V(\cdot + x)|_{I+x}\}$$

- *Finite complexity*: There are finitely many intervals $I_1, \ldots, I_n$ s.t. $\mathcal{R}_{I_i}$ is discrete and $\mathbb{R} = \bigcup_{i=1}^{n} \mathcal{R}_{I_i} + I_i$.
- *Linear Repetitivity*: Every set $\mathcal{R}_I$ is relatively dense with gaps $O(|I|)$.
- *Uniform density* + **bounded local discrepancy**: Every $\mathcal{R}_{I_i}$ there exists $\nu > 0$

$$\sup_{n \in \mathbb{Z}} |\#\mathcal{R}_{I_i} \cap [-n, n] - \nu n| < +\infty.$$

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Ex: $s_n = \lfloor n\alpha \rfloor - \lfloor (n-1)\alpha \rfloor$, with $\alpha \in (0,1)$ irrational bounded coeff. in the continued fraction (Hedlund 40'-Kesten 60').

Bon Anniversaire
Jean-Marc !!