Spectral statistics of sparse random graphs: the Thomae function and modular forms

Sergei Nechaev

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with Kirill Polovnikov (*Skoltech, Moscow*)

Rare event statistics
… and beyond

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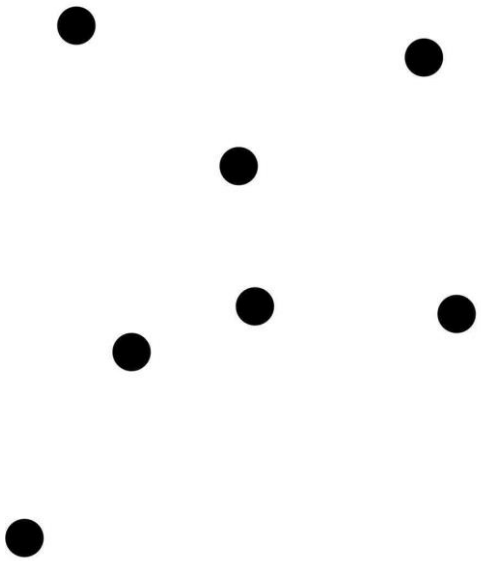
How to investigate collective properties of clusters of connected elementary units?



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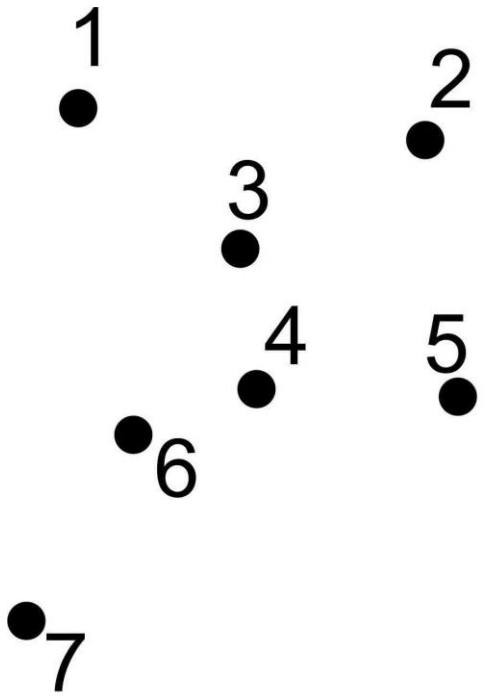
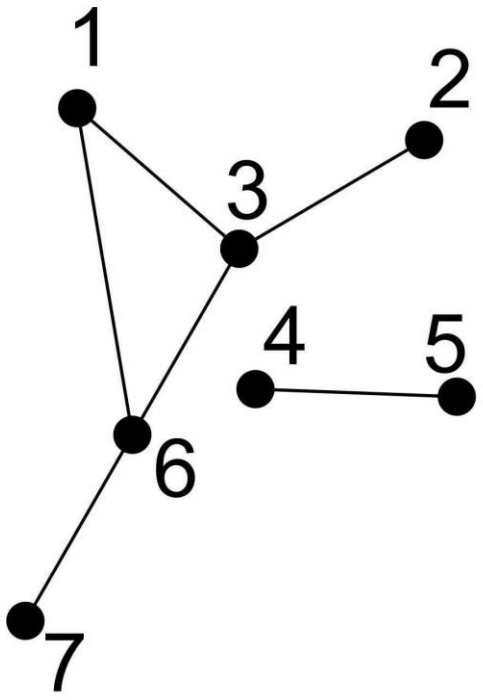


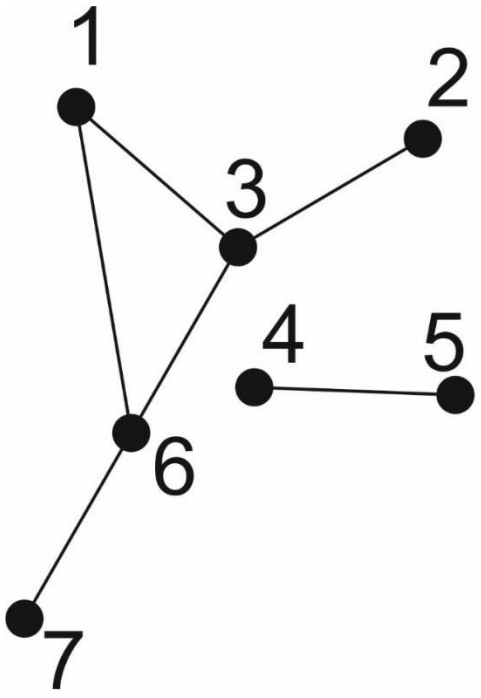
Spectrum of adjacency matrix tells us about graph topology



	1	2	3	4	5	6	7
1	0	0	1	0	0	1	0
2	0	0	1	0	0	0	0
3	1	1	0	0	0	1	0
4	0	0	0	0	1	0	0
5	0	0	0	1	0	0	0
6	1	0	1	0	0	0	1
7	0	0	0	0	0	1	0

Adjacency matrix

We are interested in the density of eigenvalues, $\rho(\lambda)$, of matrices

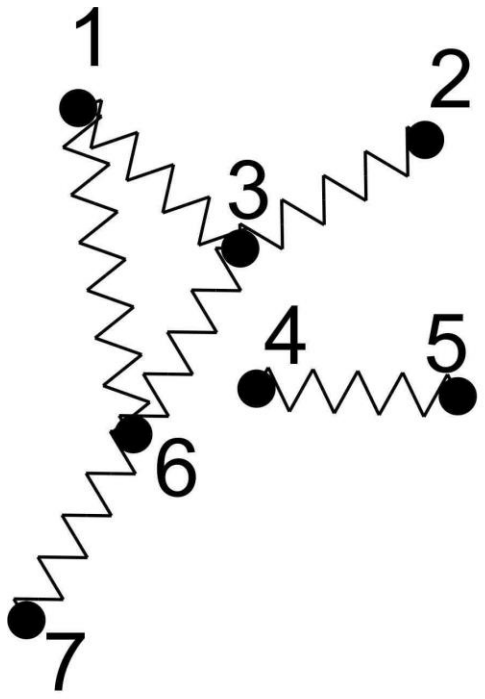


$M =$

	1	2	3	4	5	6	7
1	$-\lambda$	0	1	0	0	1	0
2	0	$-\lambda$	1	0	0	0	0
3	1	1	$-\lambda$	0	0	1	0
4	0	0	0	$-\lambda$	1	0	0
5	0	0	0	1	$-\lambda$	0	0
6	1	0	1	0	0	$-\lambda$	1
7	0	0	0	0	0	1	$-\lambda$

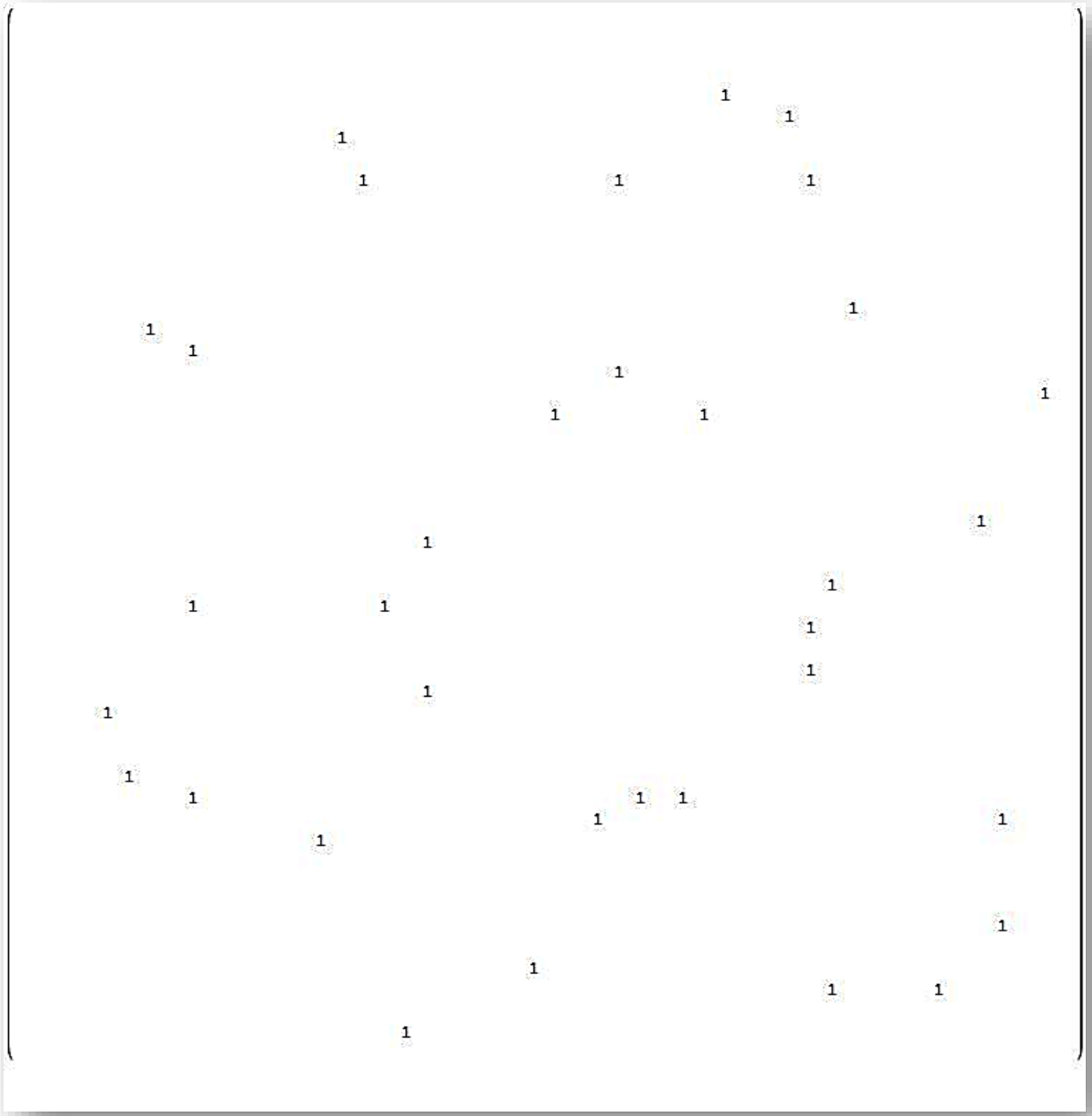
$$\det M\,(\lambda) = 0$$

Spectrum of Laplacian matrix is the spectrum of resonance frequencies of the system



	1	2	3	4	5	6	7
1	2	0	-1	0	0	-1	0
2	0	1	-1	0	0	0	0
3	-1	-1	3	0	0	-1	0
4	0	0	0	1	-1	0	0
5	0	0	0	-1	1	0	0
6	-1	0	-1	0	0	3	-1
7	0	0	0	0	0	-1	1

Laplacian matrix



$$a_{ij} = \begin{cases} 1 & \text{prob } r \\ 0 & \text{prob } 1-r \end{cases}$$

$$N \times N = 50 \times 50$$

$r = \dfrac{1}{N}$, corresponds to percolation (N >>1)

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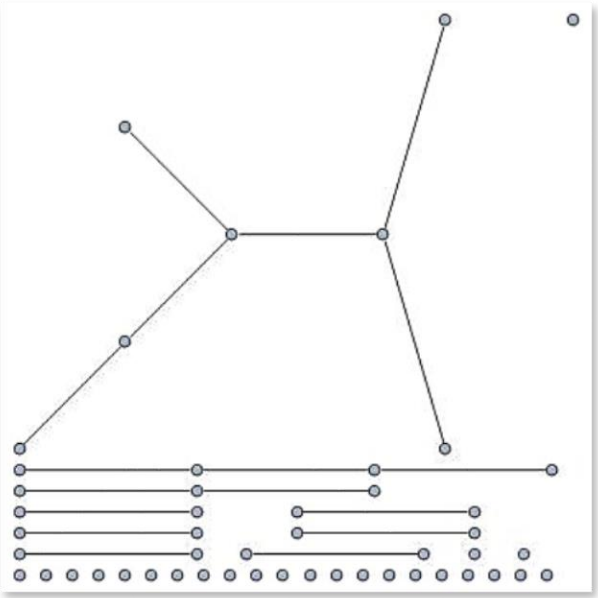

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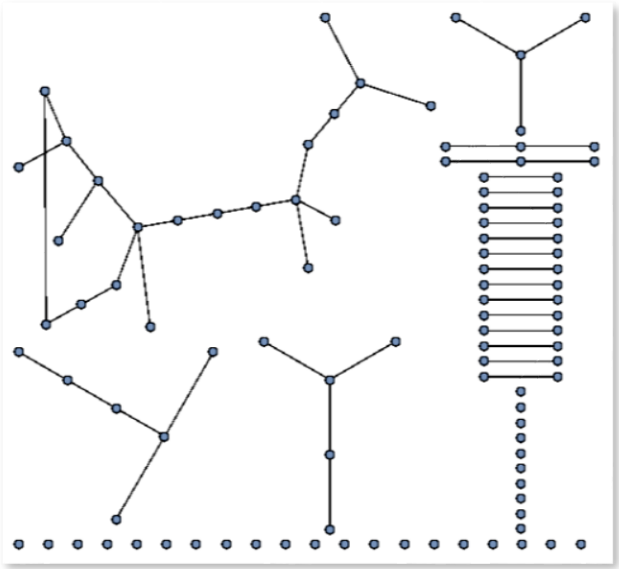
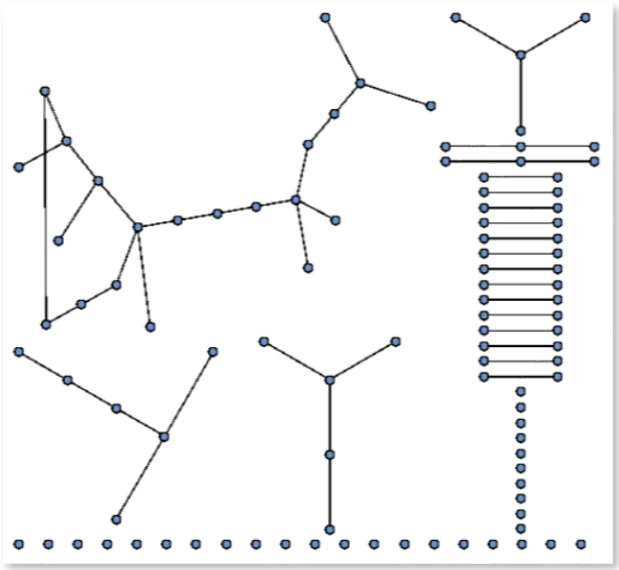
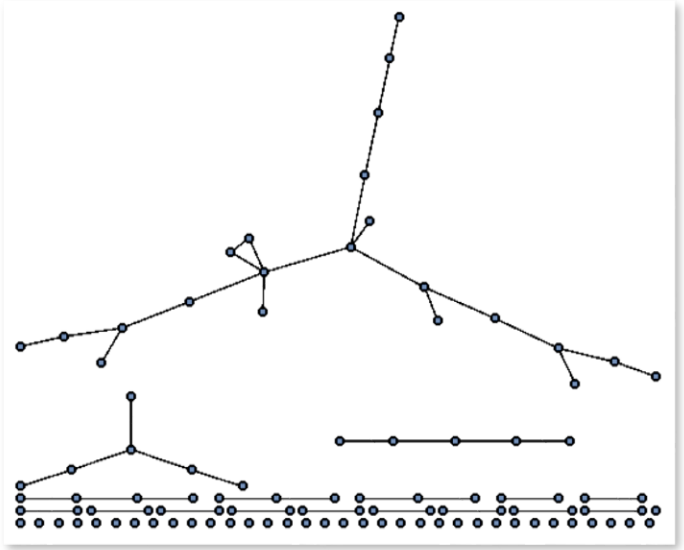
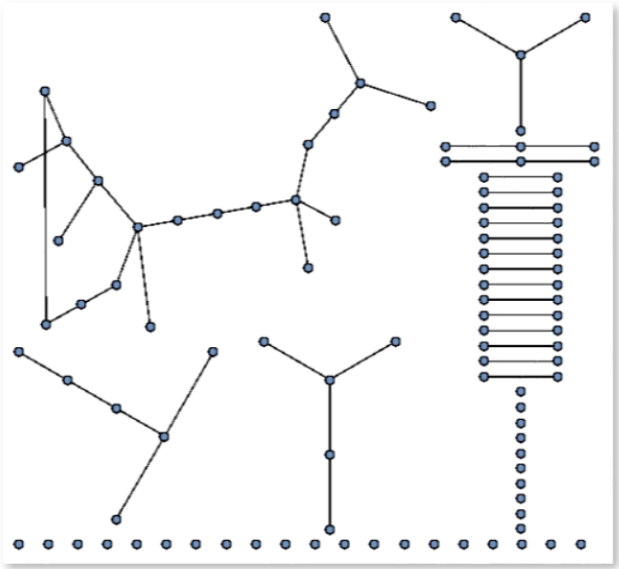
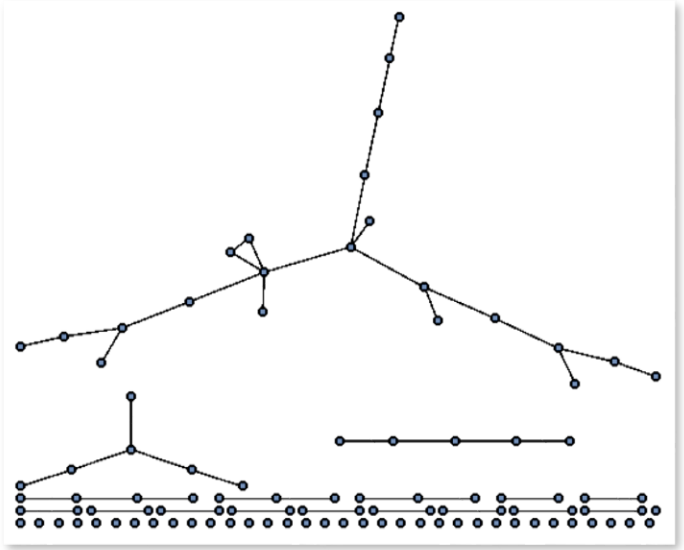
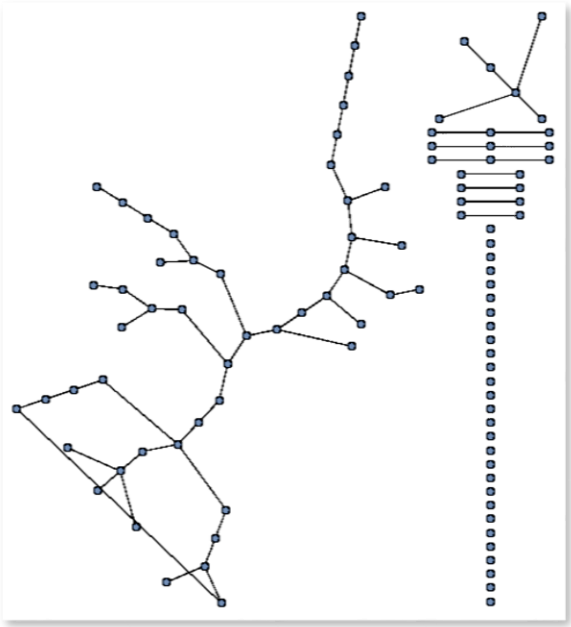


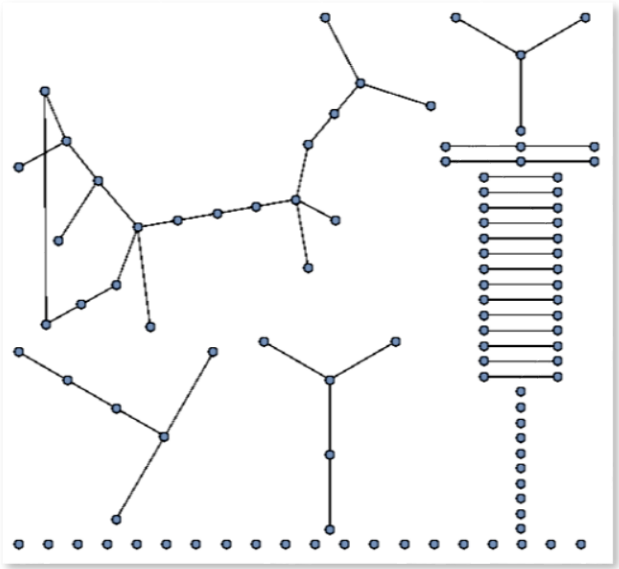


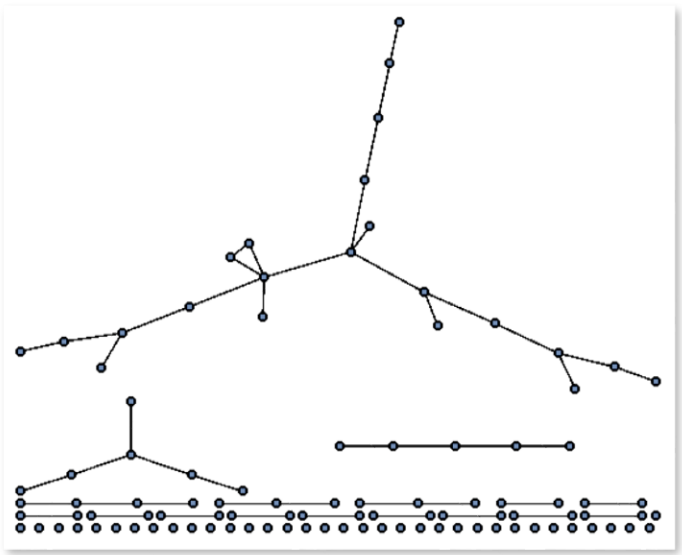


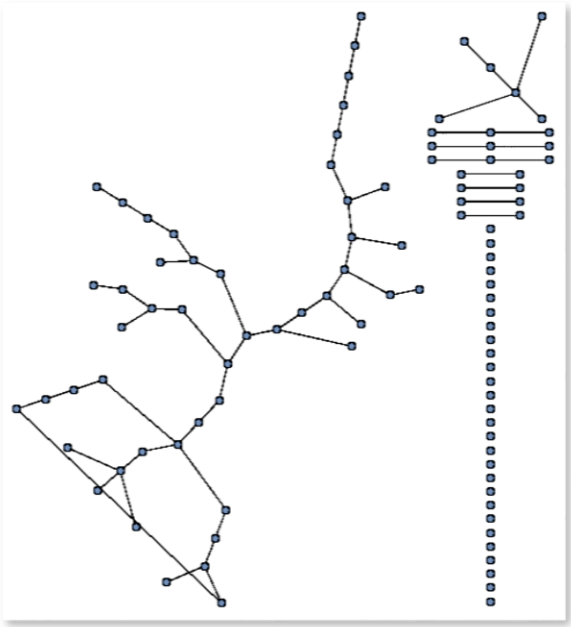


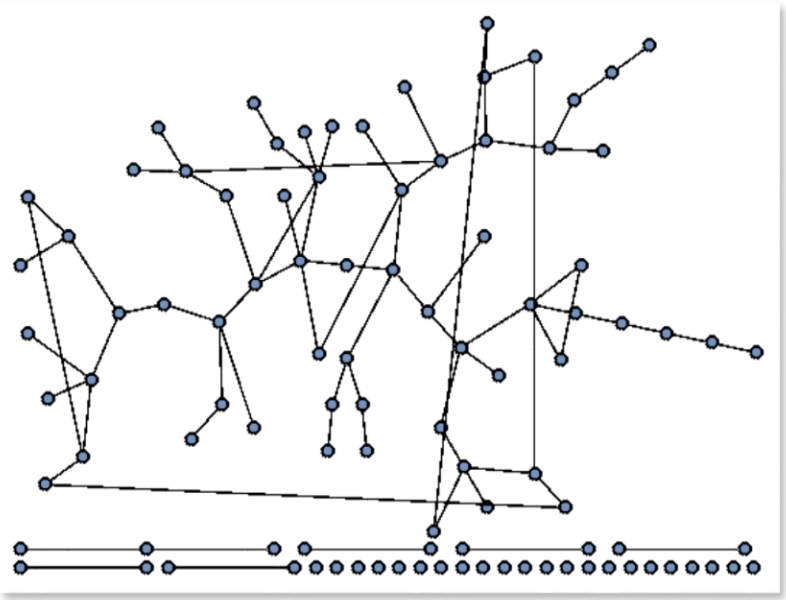


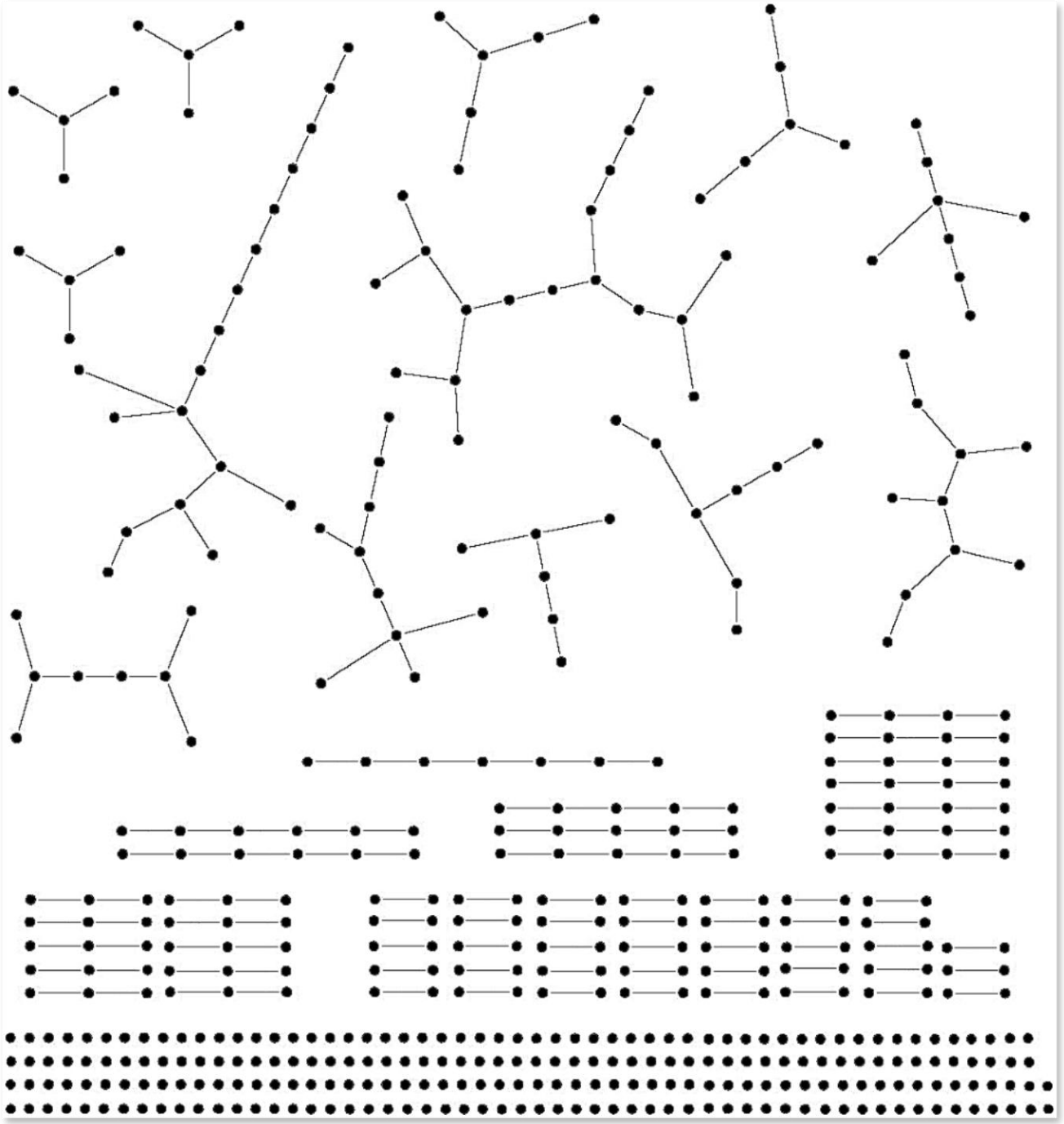








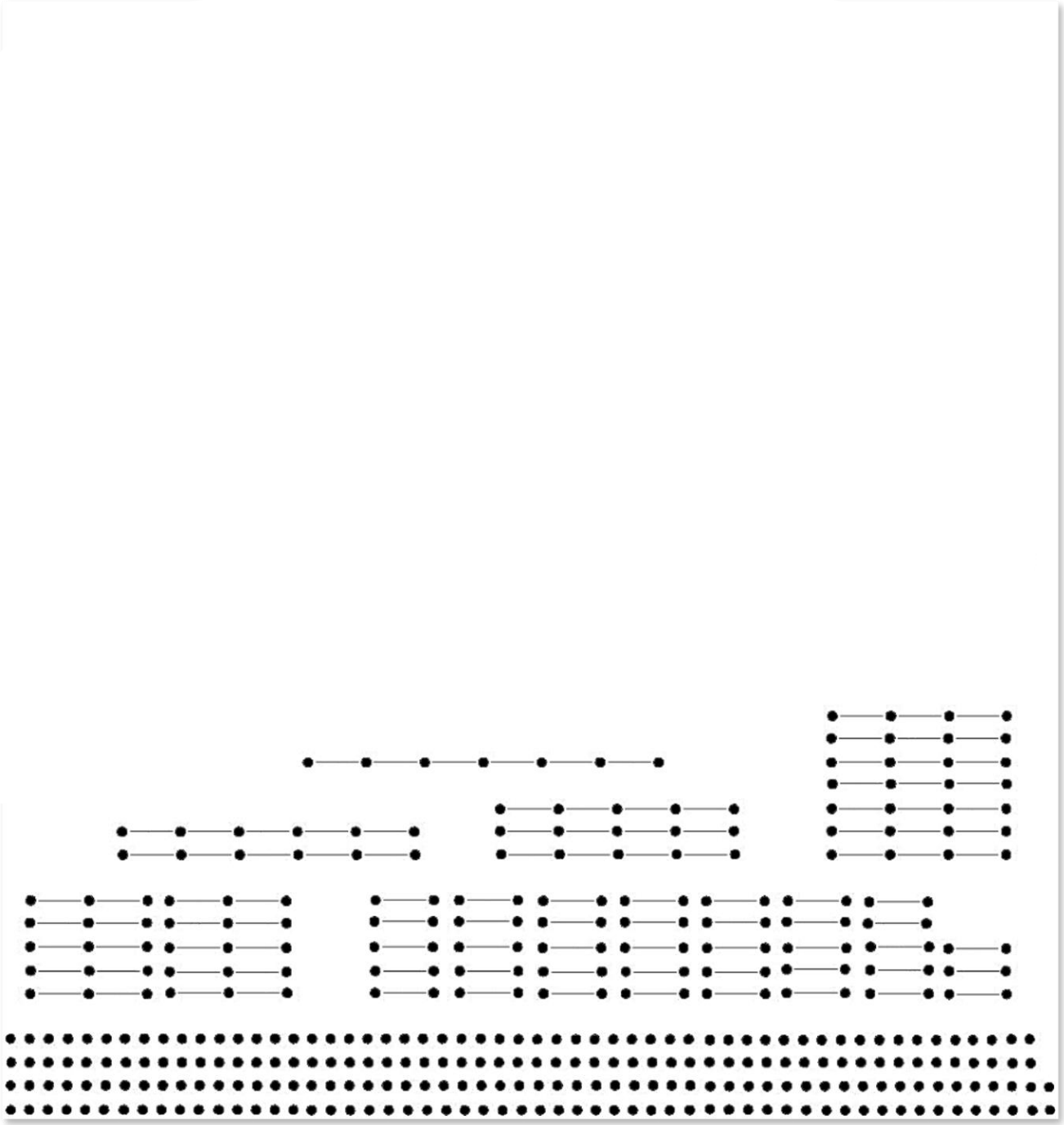




At percolation

$r = 1/N$

$(N = 500)$



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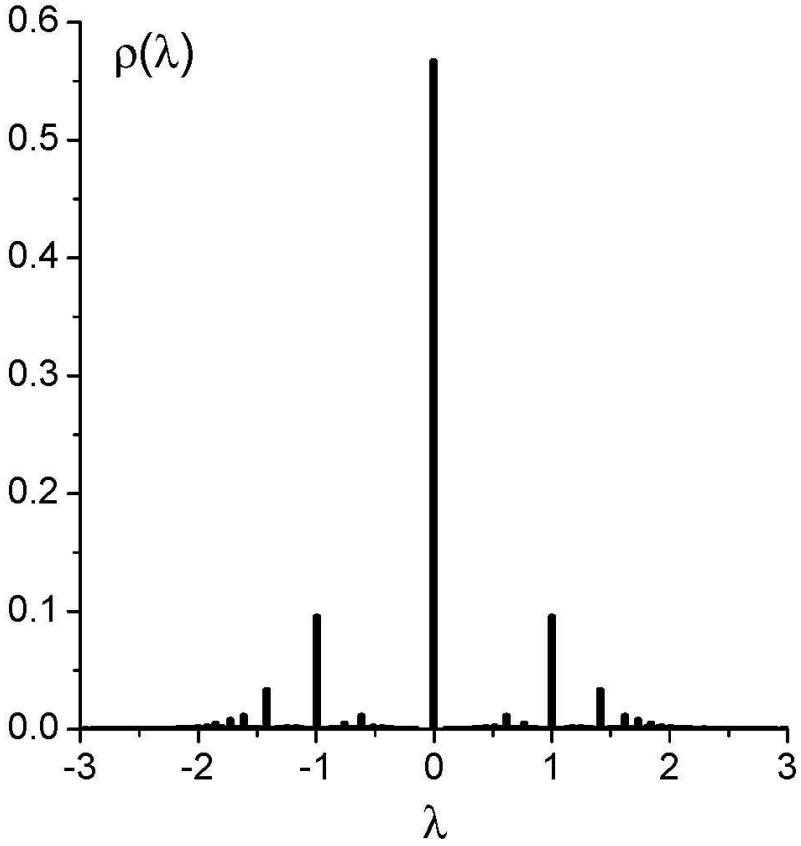
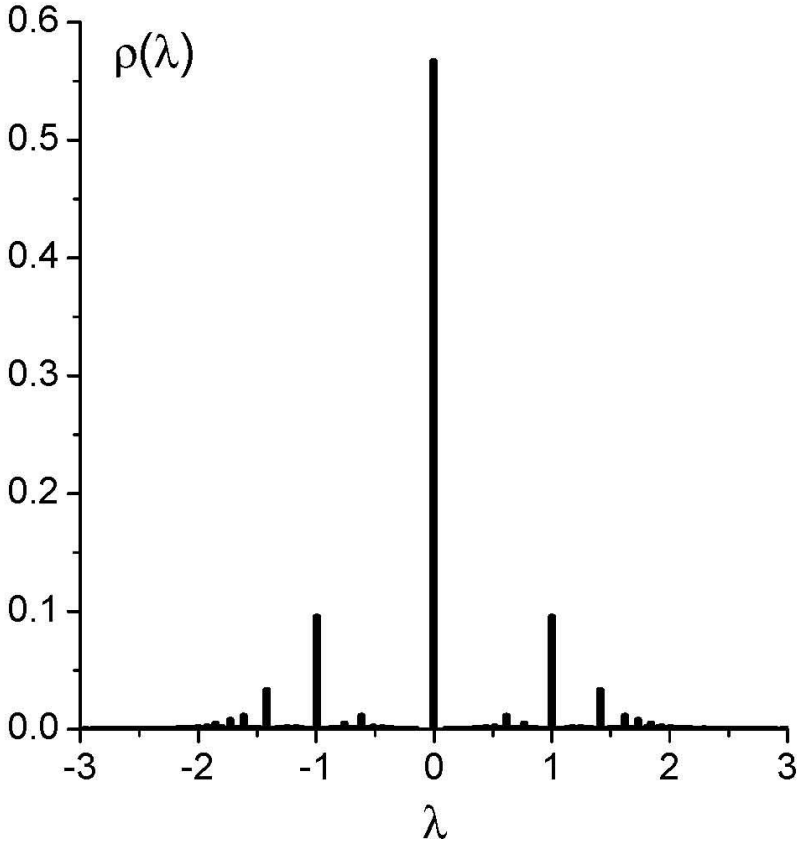
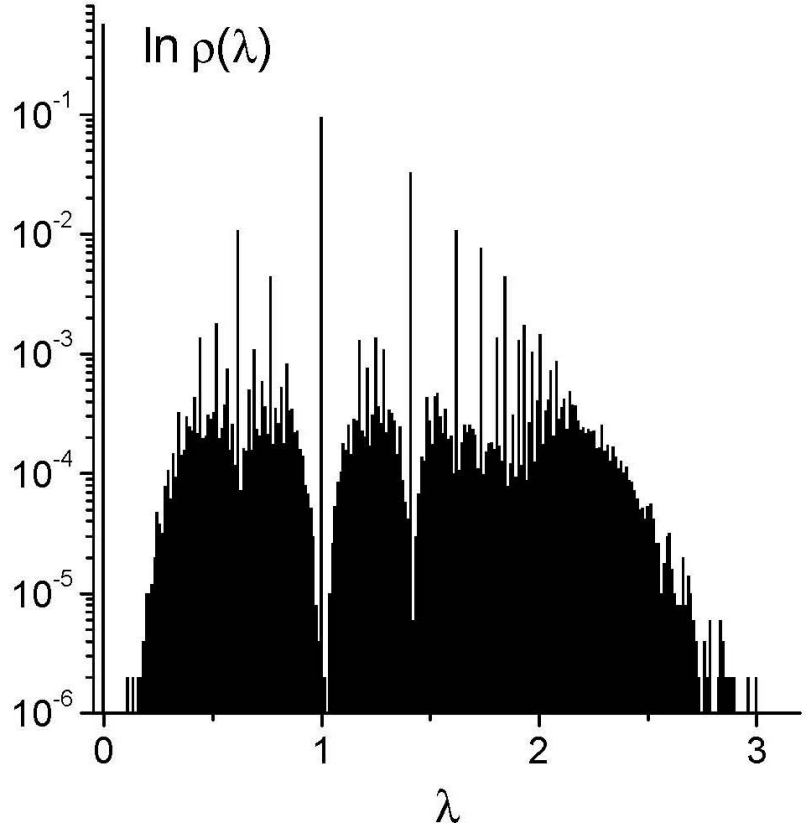


Spectral density $\rho(\lambda)$ of sparse random graphs at percolation point



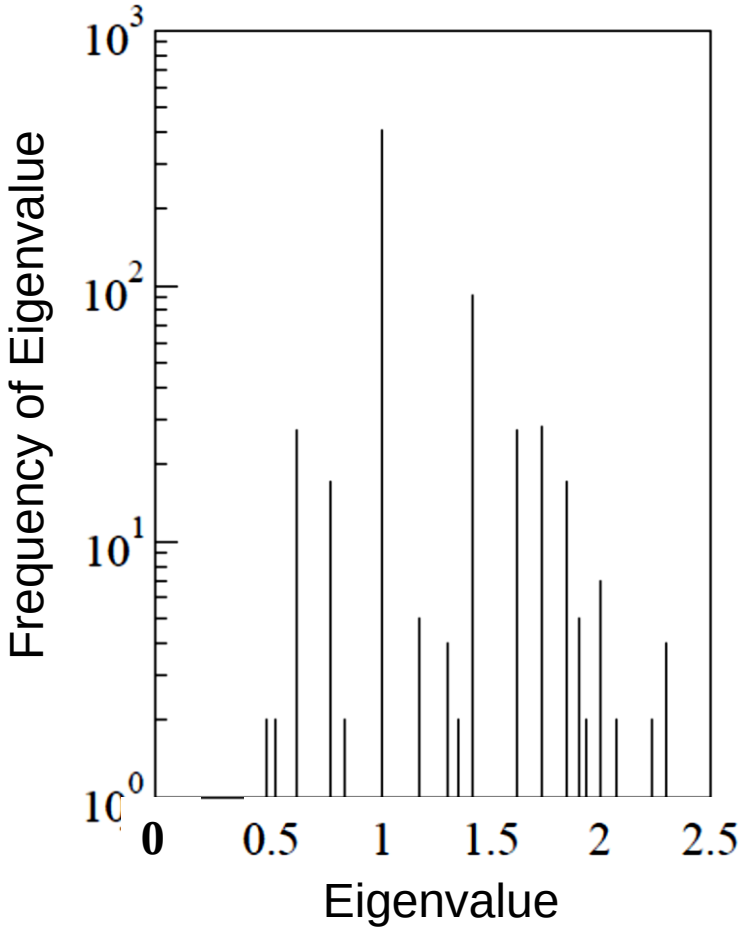


Semi-logarithmic plot for $\lambda > 0$

Spectral density $\rho(\lambda)$ of sparse random graphs at percolation point

Spectral statistics of protein-protein interaction network in *Drosophyla melanogaster*

C. Kamp, K. Christensen, Phys. Rev. E (2005)



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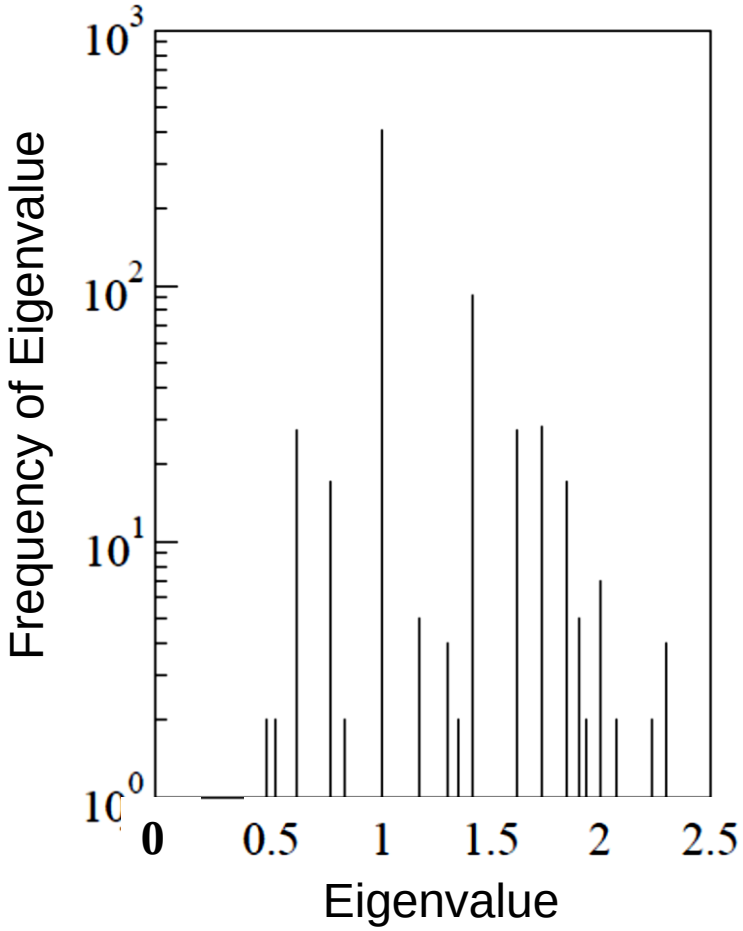
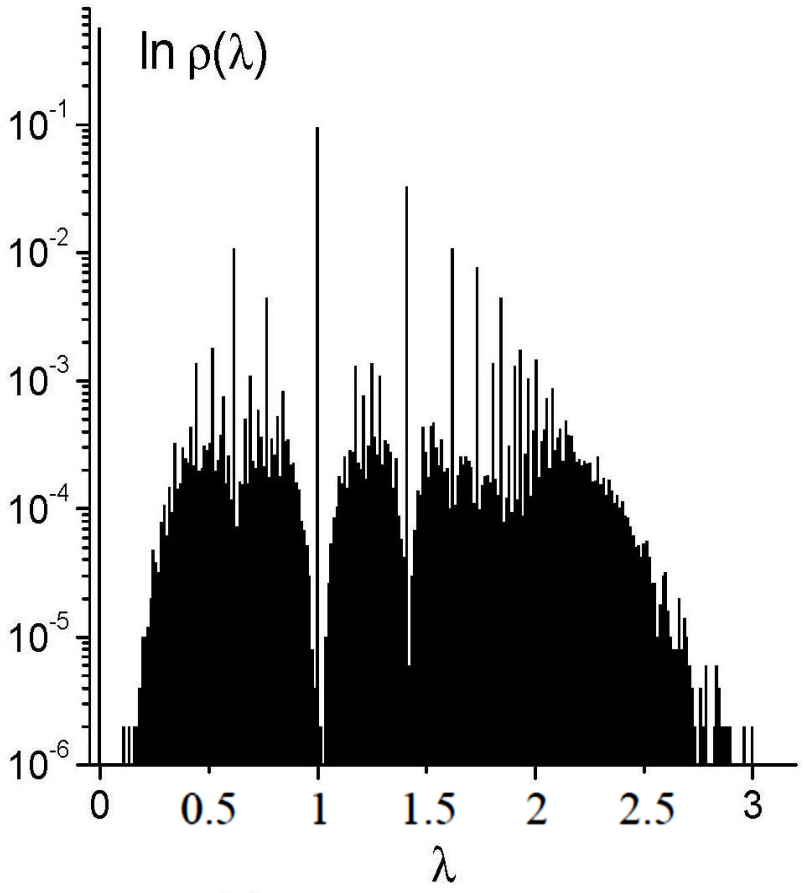
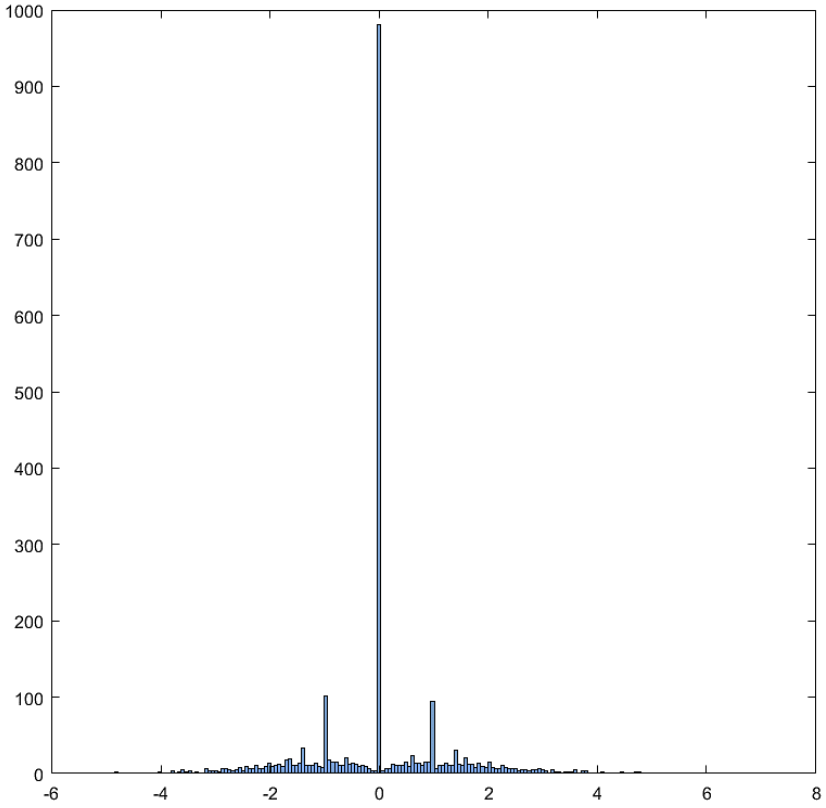




Sample of spectral statistics of adjacency matrix of X chromosome in single-cell experiments (resolution 10 kb)



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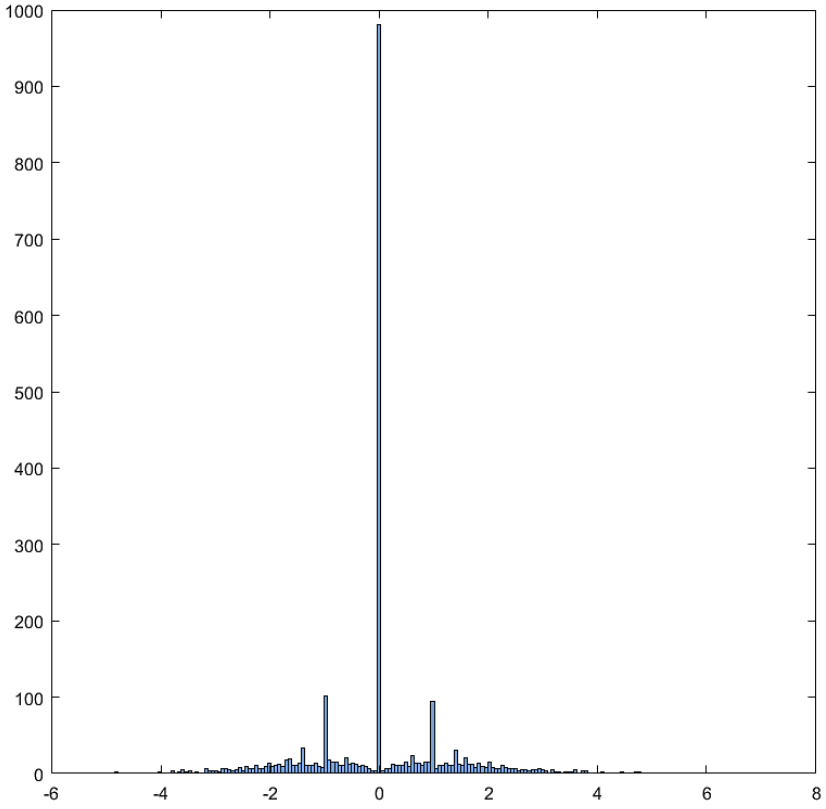
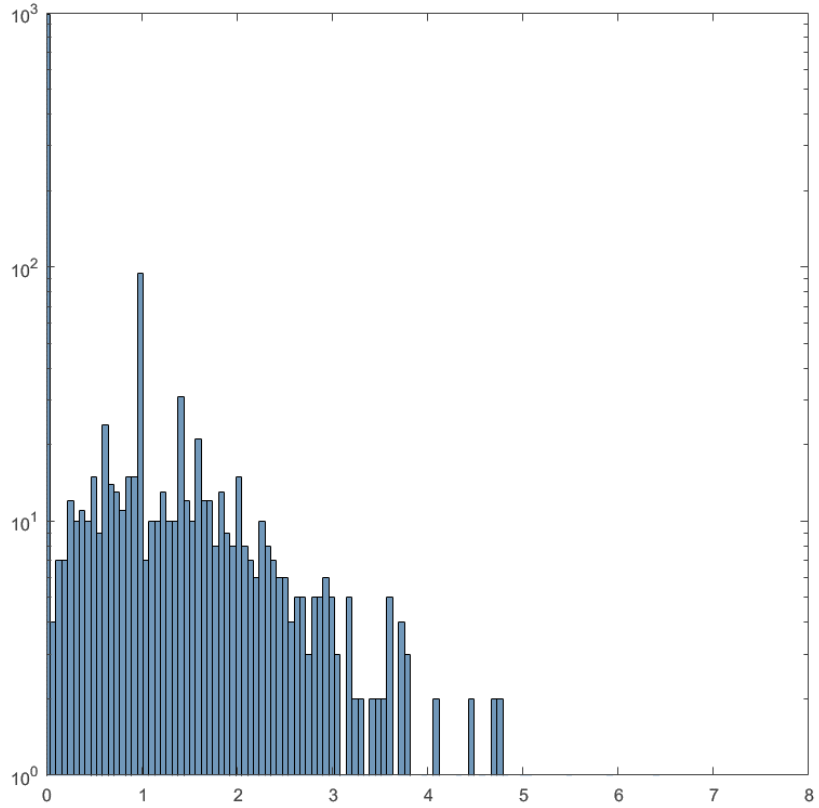
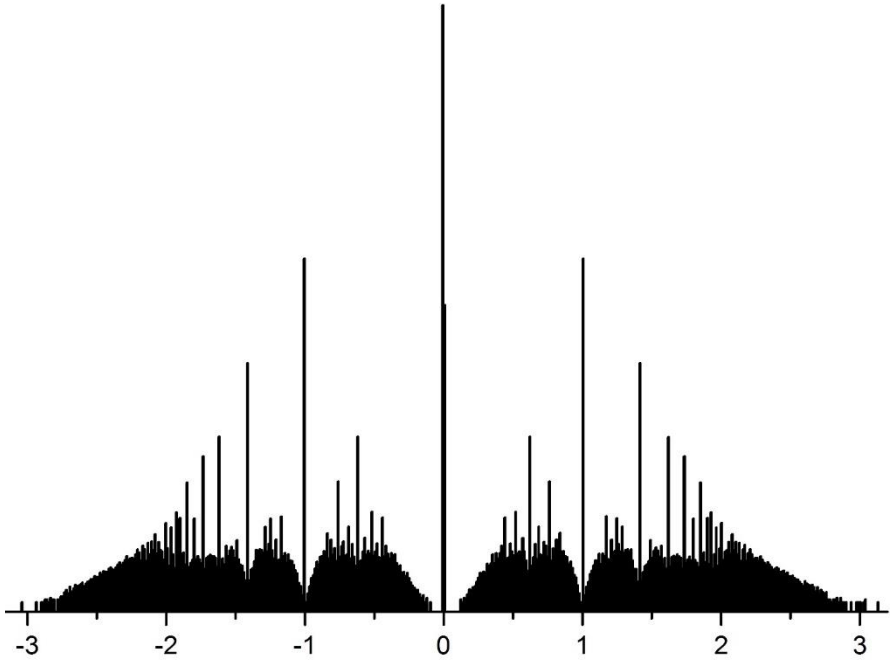
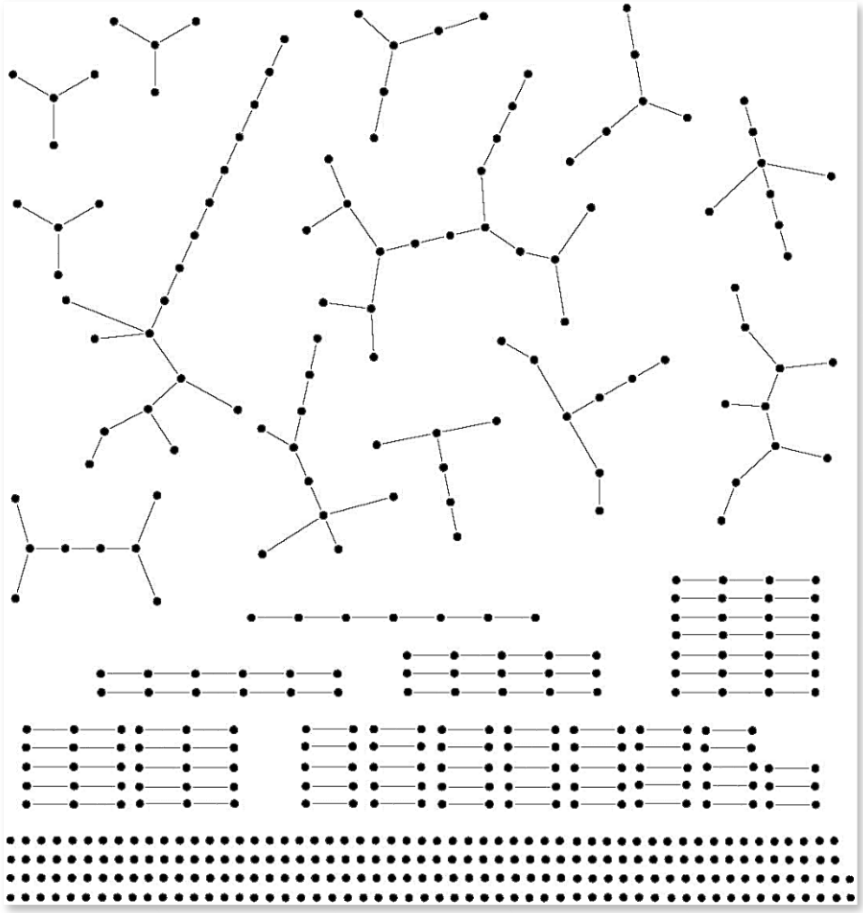




Semi-logarithmic plot

We have computed the distribution of **all** clusters in sizes, and separately – of clusters of **linear chains** only





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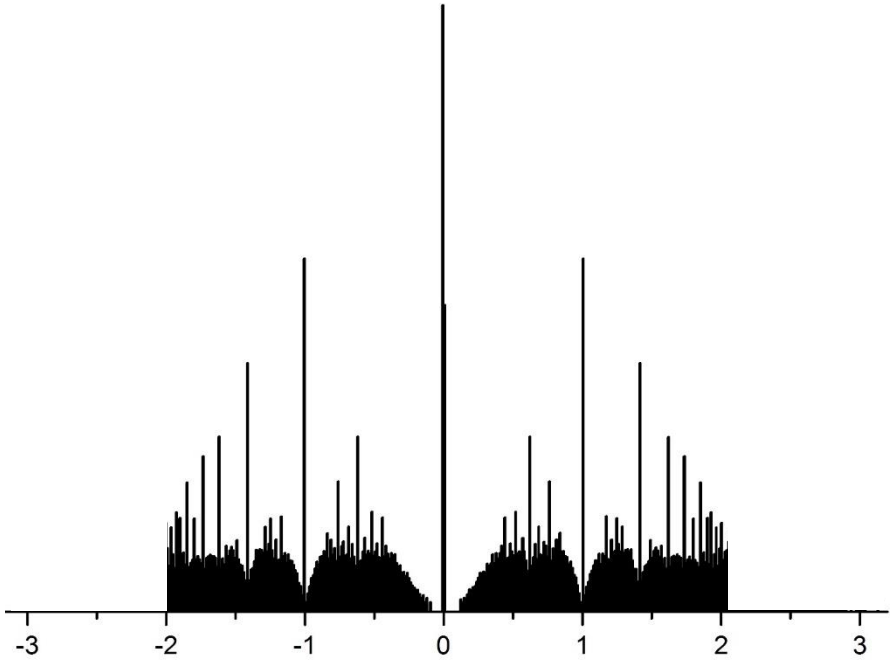

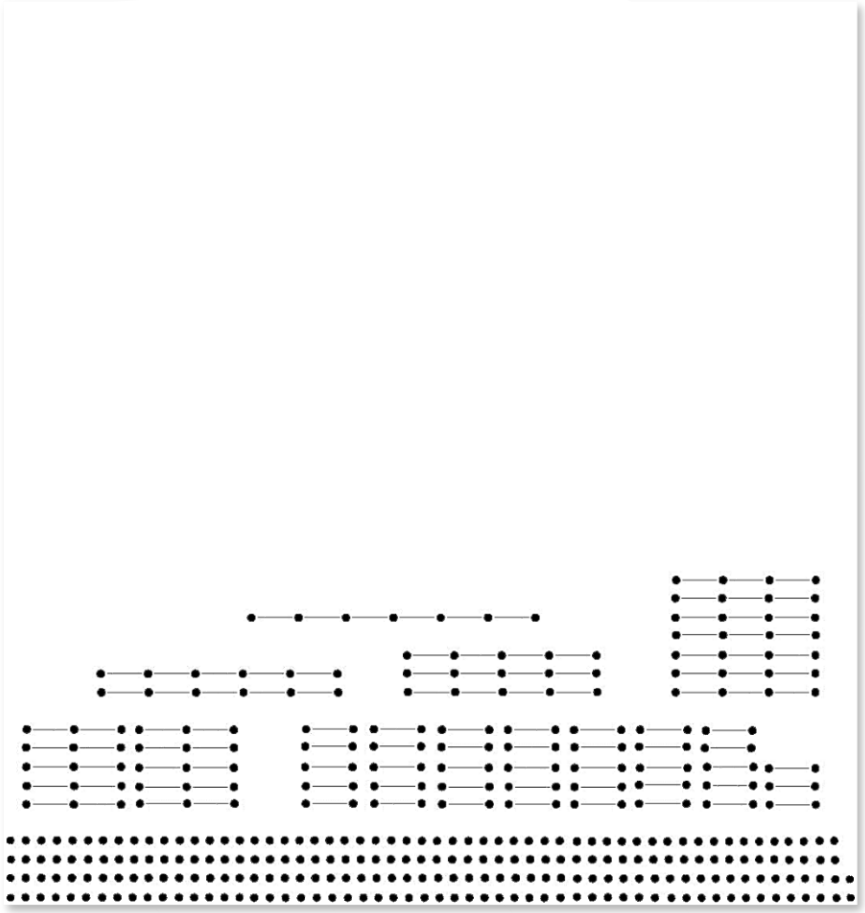

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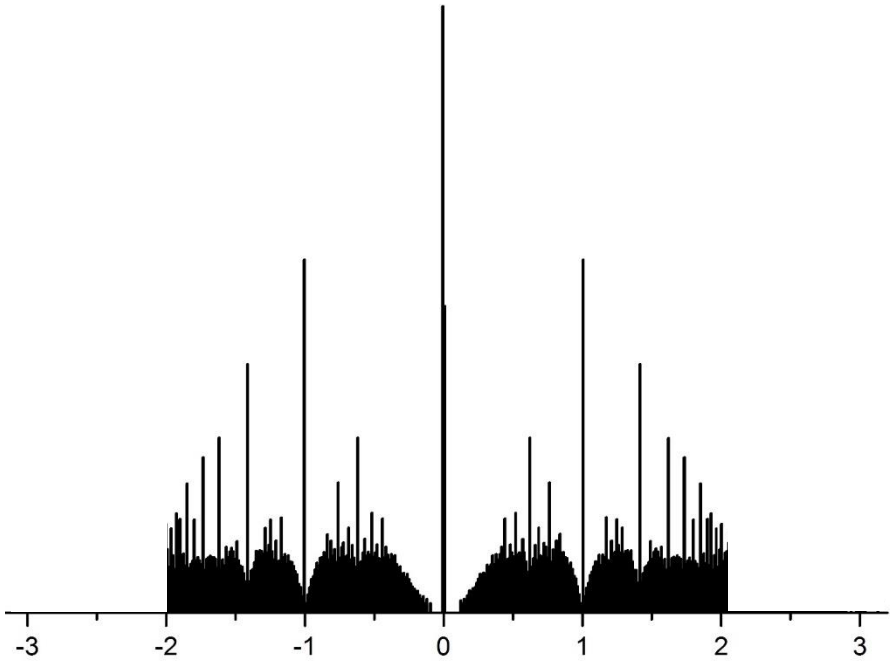
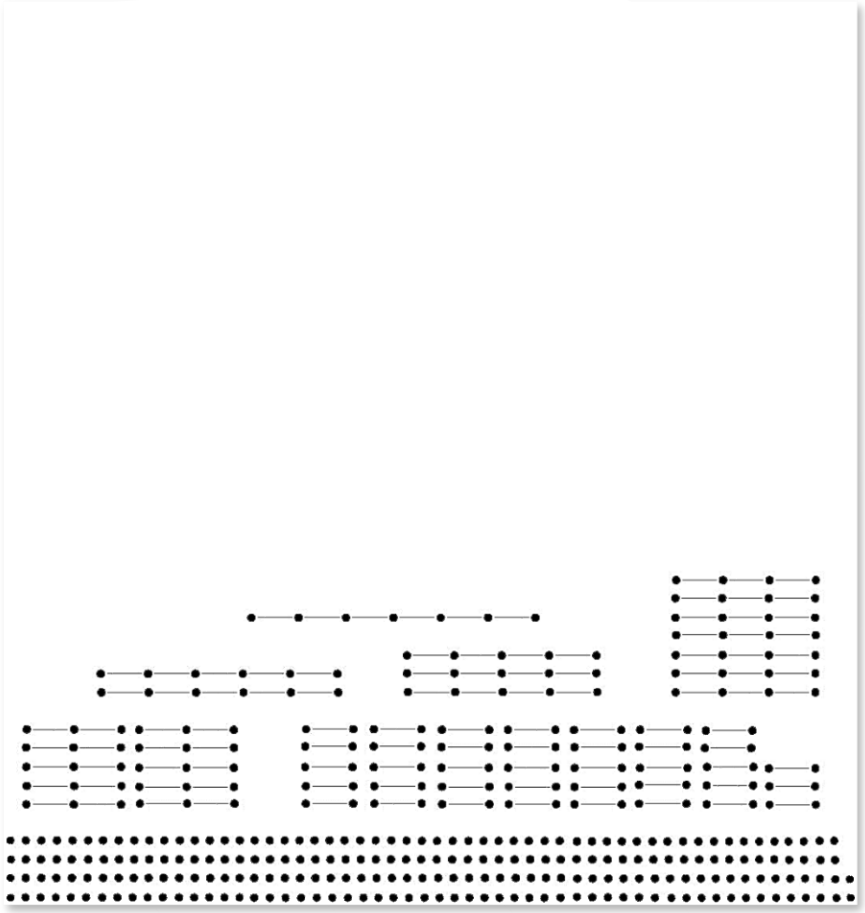




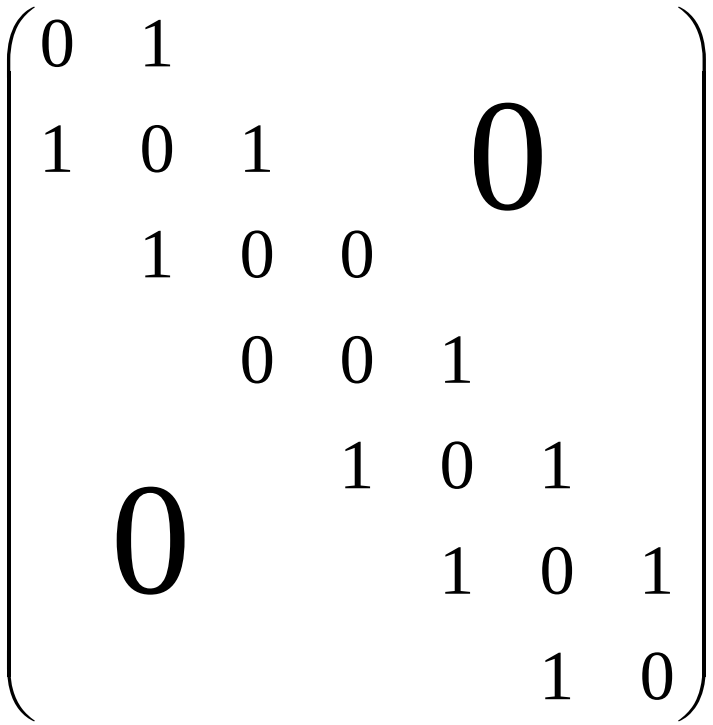
$\left(\frac{2-e^{-1}}{e-1}\right)\times 100\% \approx 95\%$ of all random subgraphs at percolation are linear “polymers” with distribution $P(n) \sim e^{-n}$

Consider an ensemble of two(three)-diagonal matrices

$$\begin{pmatrix} 0 & x_1 & 0 & 0 & \cdots \\ x_1 & 0 & x_2 & 0 & \\ 0 & x_2 & 0 & x_3 & \\ 0 & 0 & x_3 & 0 & \\ \vdots & & & & \ddots \end{pmatrix}$$

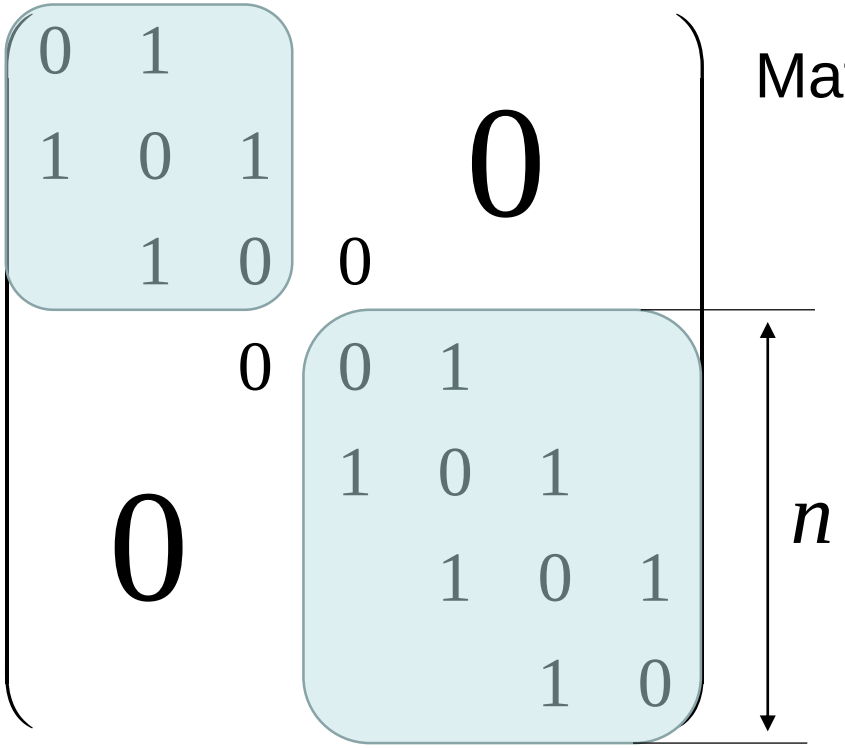
where the matrix elements are

$$x_i = \begin{cases} 1 & \text{with probability } q \\ 0 & \text{with probability } 1-q \end{cases}$$



Matrix N x N (N >>1)

Adjacency matrix spits in uniform Jordan cells with the distribution ~ q^n ($0<q<1$) in sizes



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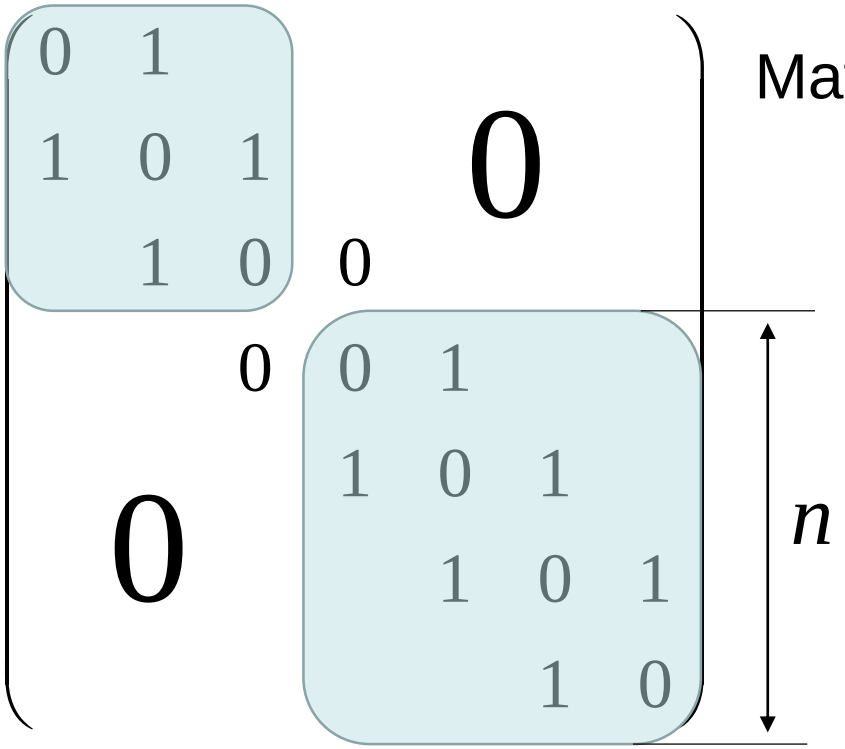

Matrix N x N (N >>1)

The set of eigenvalues in the cell of size n x n is

$$\lambda_k = 2\cos\frac{\pi k}{n+1}; \qquad (k = 1, \ldots, n)$$

Spectral density, $\rho_{\text{lin}}(\lambda)$, of the ensemble of random matrices is:

$$\rho_{\text{lin}}(\lambda) = \lim_{N\to\infty} \frac{1}{N} \left\langle \sum_{k=1}^{N} \delta(\lambda - \lambda_k) \right\rangle$$

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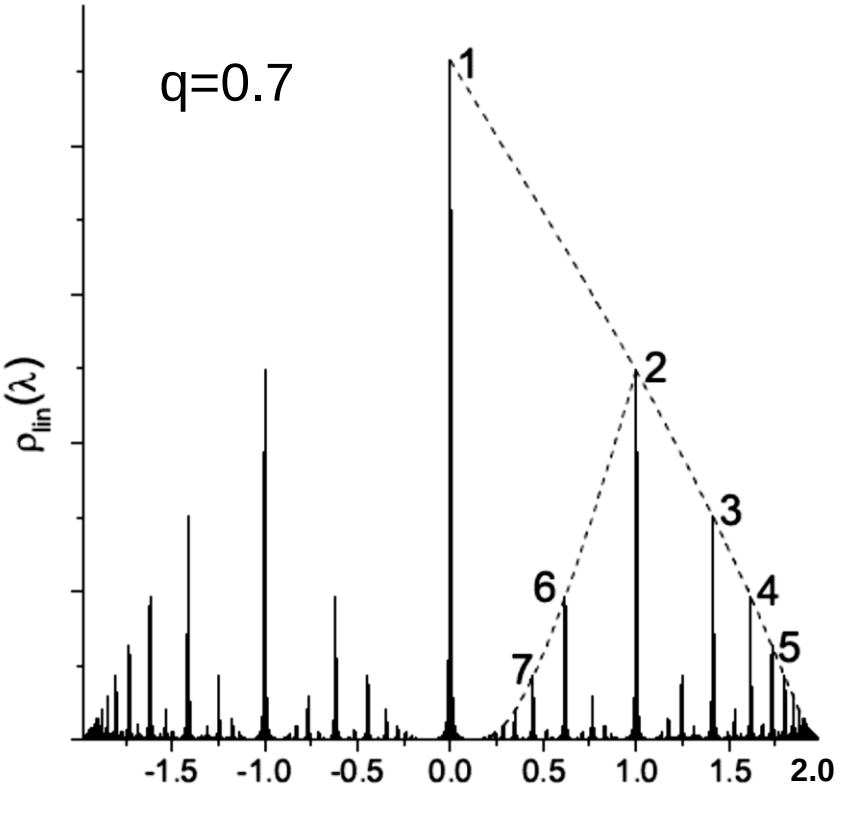
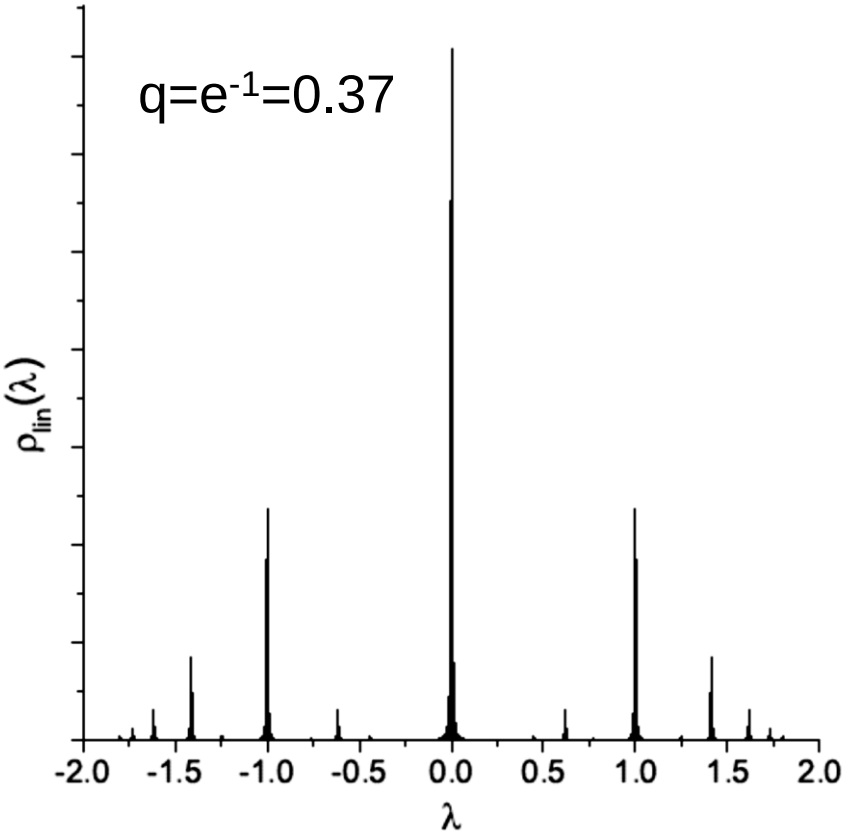
Summing the spectra of exponentially weighted Jordan cells, we get:

$$\rho_{\text{lin}}(\lambda) = \lim_{\substack{N\to\infty \\ \varepsilon\to 0}} \frac{\varepsilon}{\pi N} \sum_{n=1}^{N} q^n \sum_{k=1}^{n} \frac{1}{\left(\lambda - 2\cos\frac{\pi k}{n+1}\right)^2 + \varepsilon^2}$$

Spectral density of an ensemble of random three-diagonal operators





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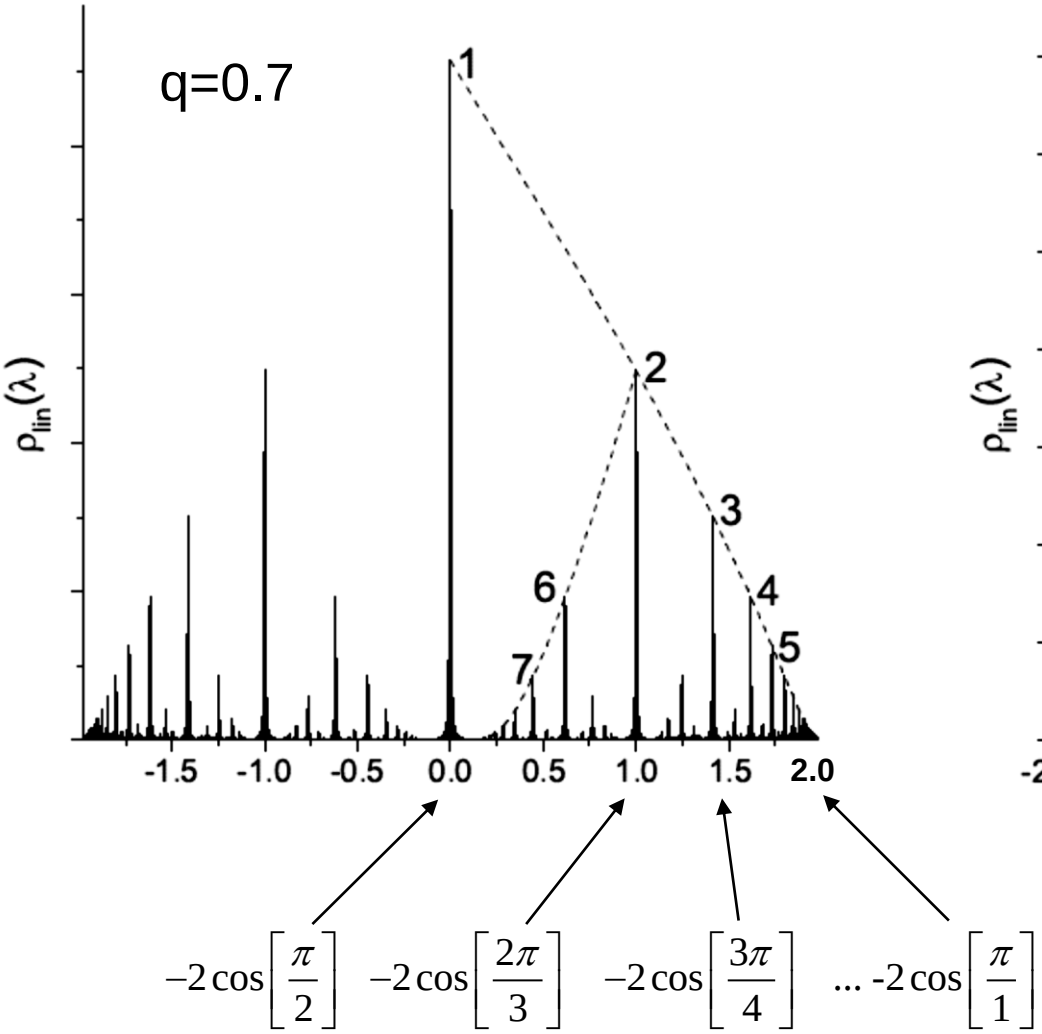

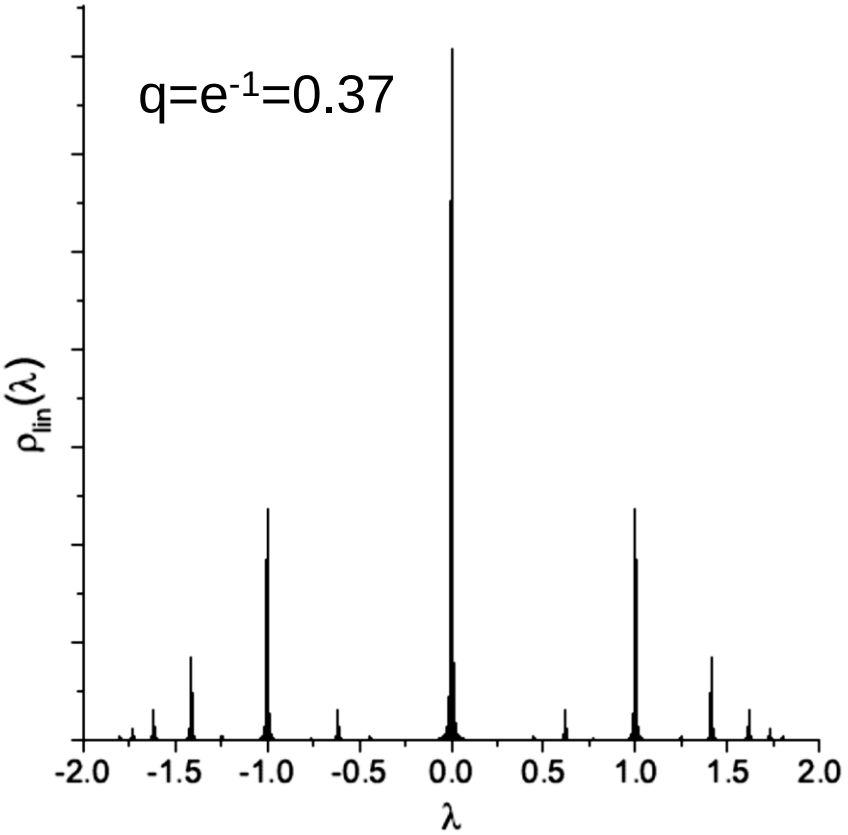

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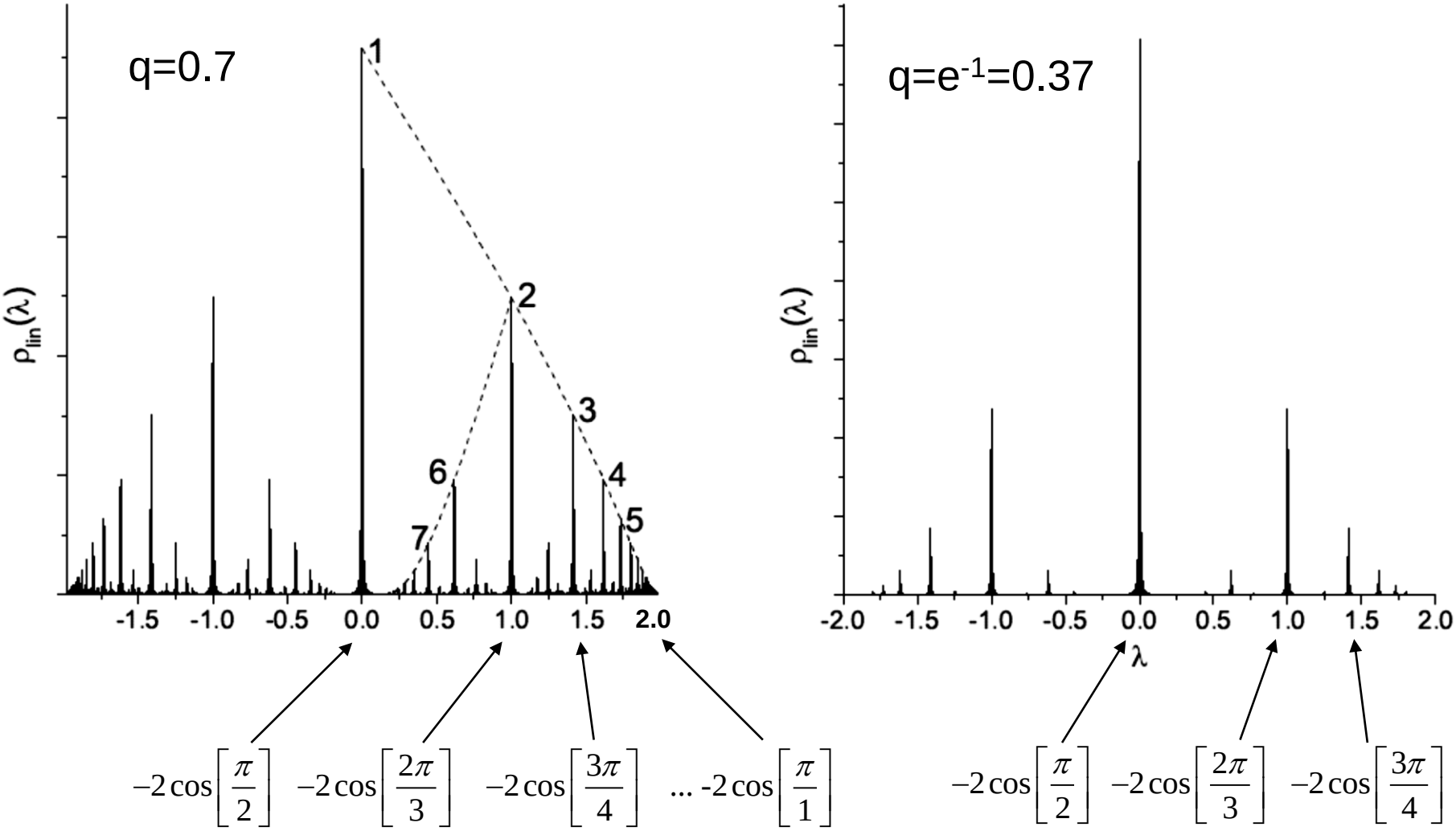
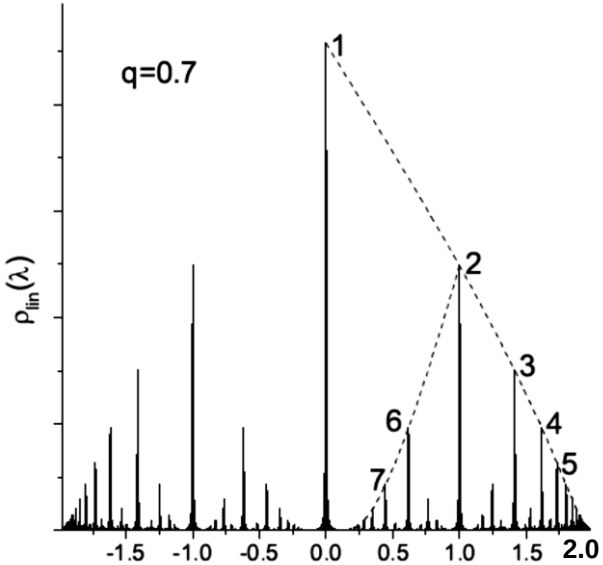


Positions of resonances of the principle series of peaks (the enveloping curve)



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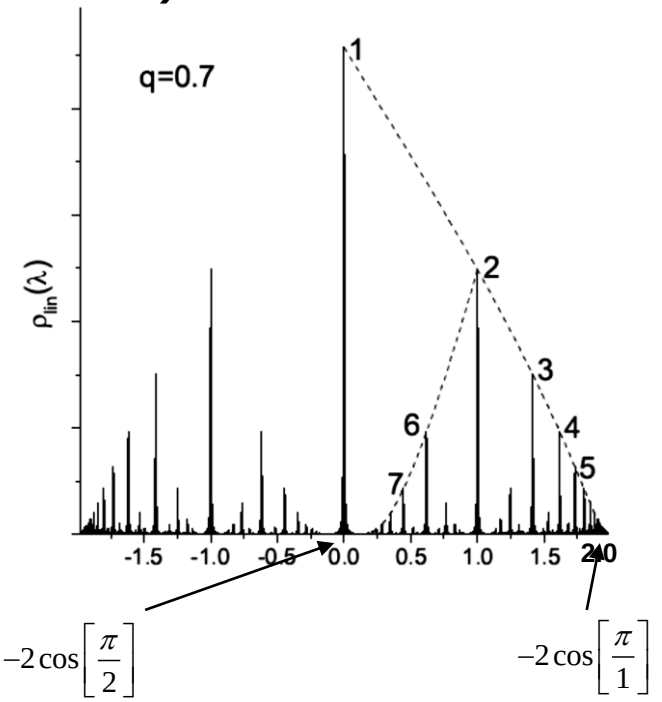

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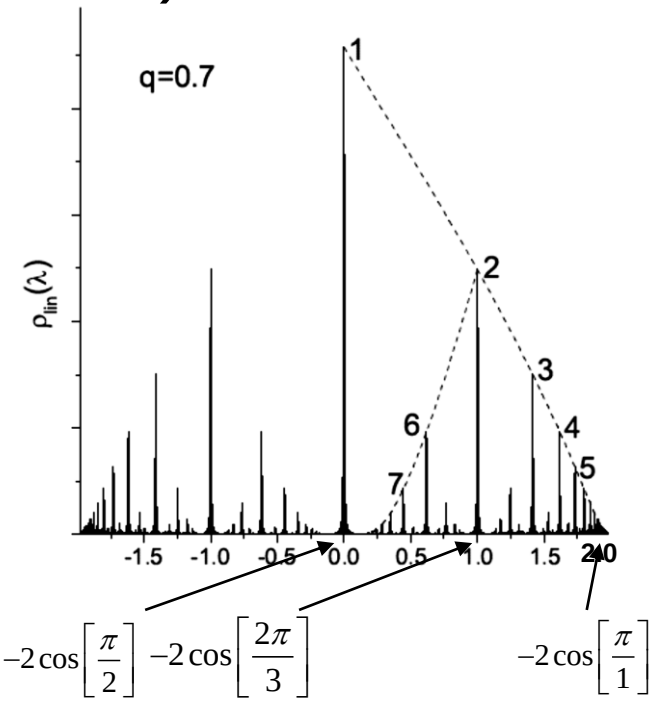
$$\lambda_2 = -2\cos\left[\pi\left(\frac{1}{2}\oplus\frac{1}{1}\right)\right] = -2\cos\frac{2\pi}{3} = 1.$$



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$$\lambda_1 \equiv -2\cos\frac{\pi}{2} = 0, \quad \lambda_\infty \equiv -2\cos\frac{\pi}{1} = 2$$

$$\lambda_2 = -2\cos\left[\pi\left(\frac{1}{2}\oplus\frac{1}{1}\right)\right] = -2\cos\frac{2\pi}{3} = 1$$

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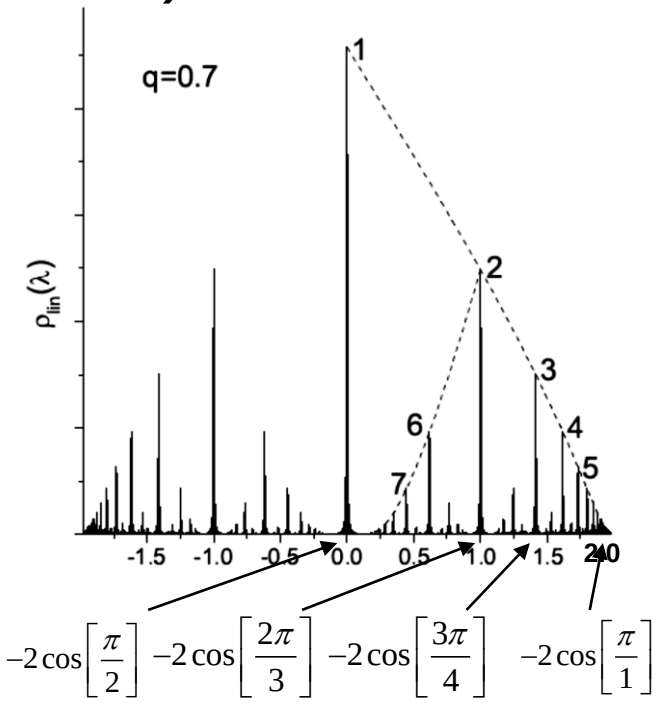

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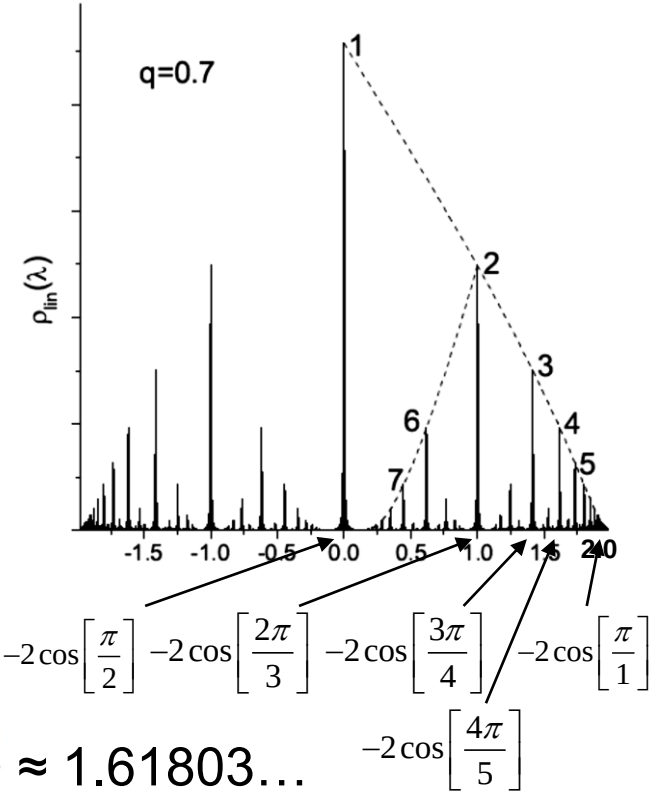
(golden mean)



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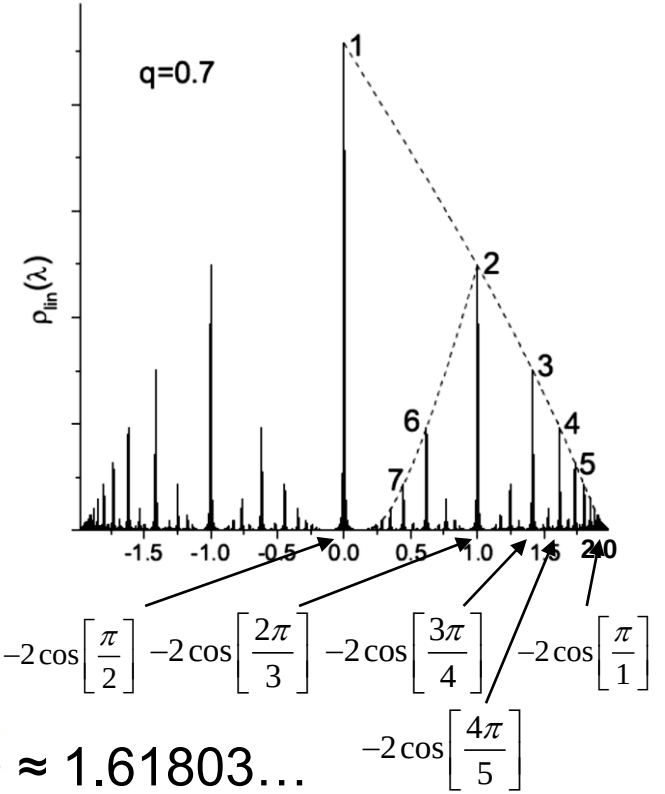
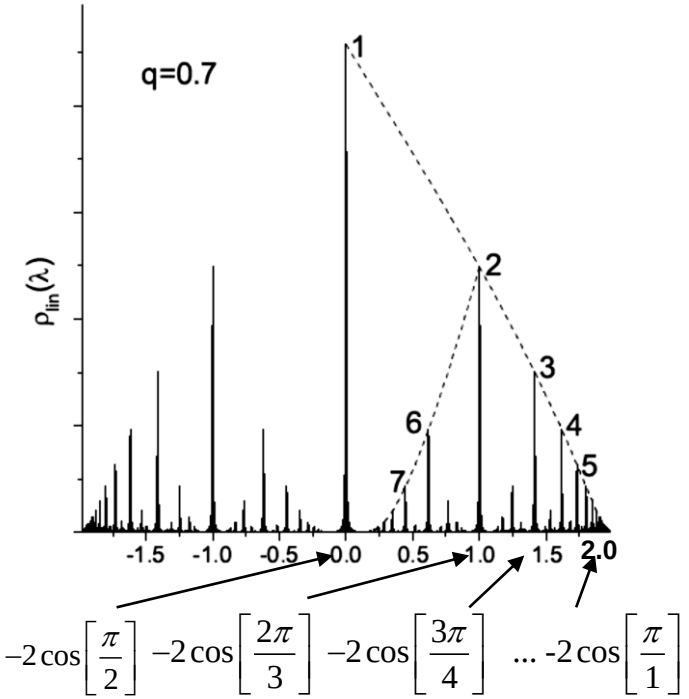
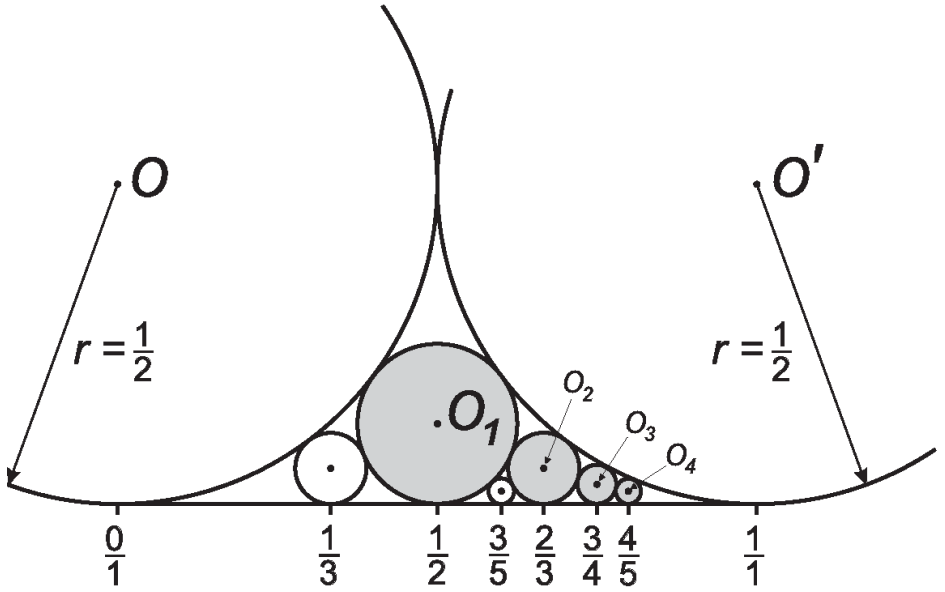


Generic expression for the principle series of peaks

$$\lambda_k = -2\cos\frac{k\pi}{k+1}, \quad (k = 1, 2, \ldots)$$

Positions of resonances is determined by Farey numbers





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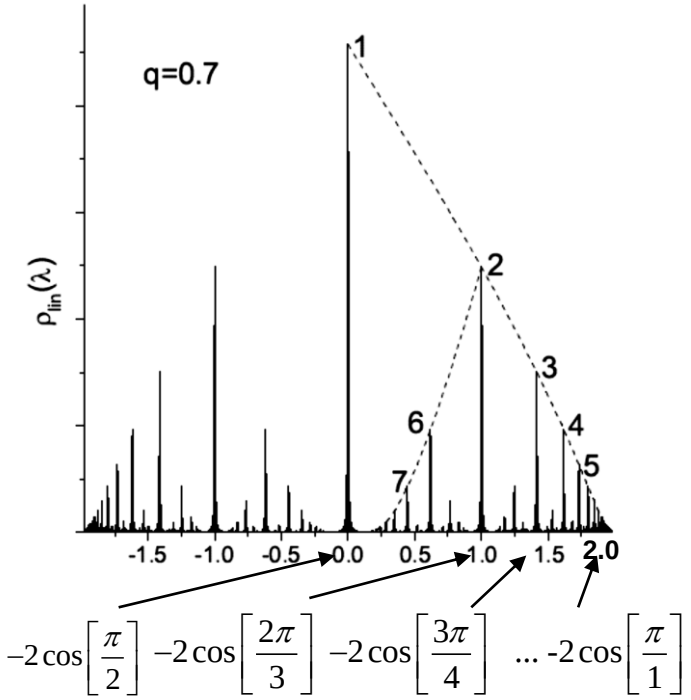

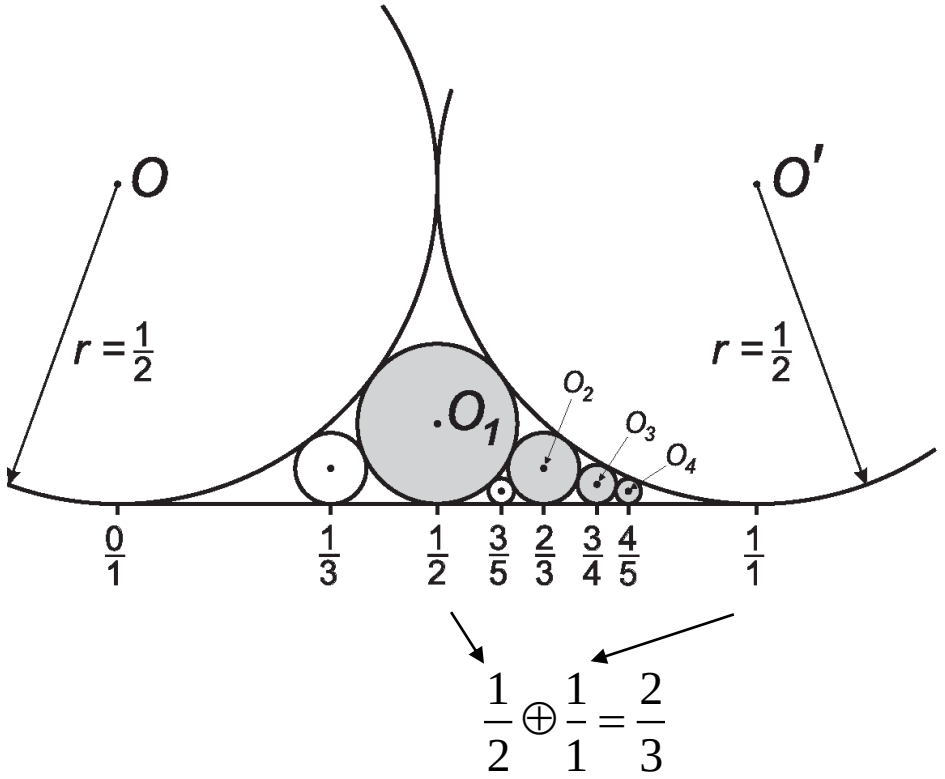

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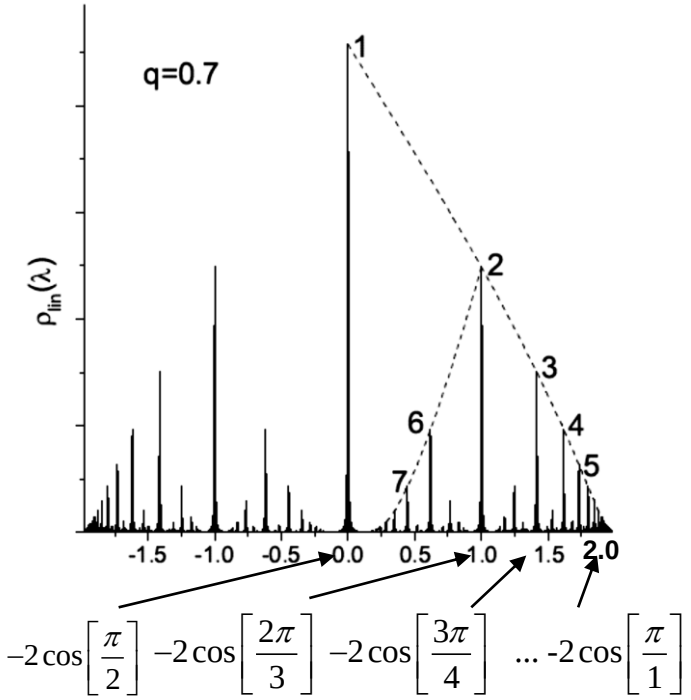

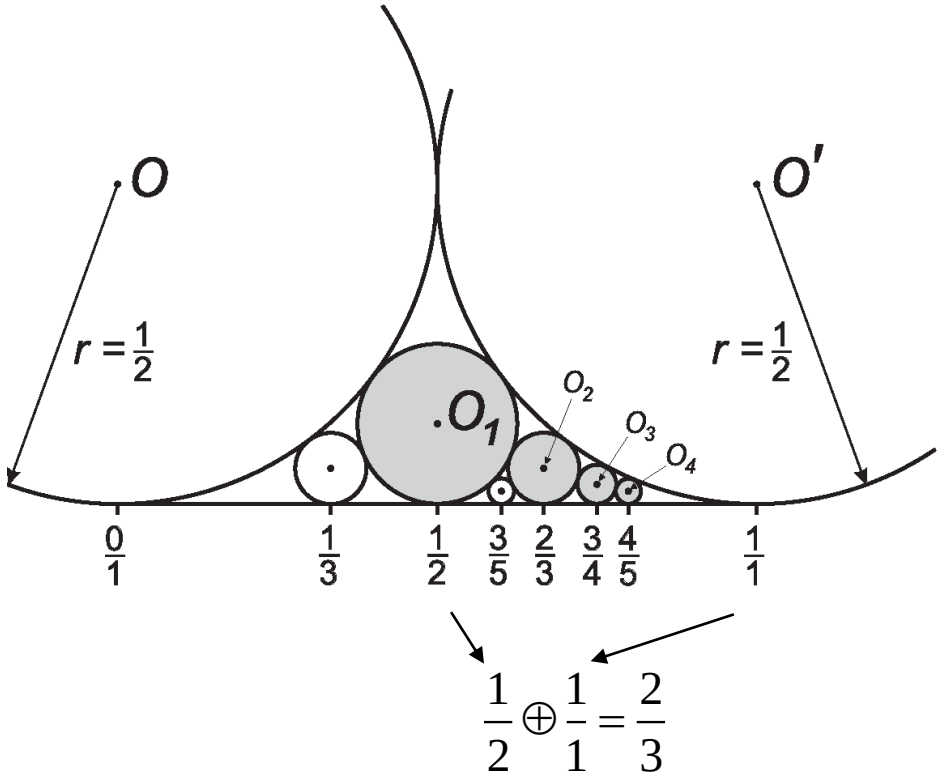

$$\frac{p_n}{q_n}=\frac{p_{n-1}}{q_{n-1}}\oplus\frac{p_{n+1}}{q_{n+1}}=\frac{p_{n-1}+p_{n+1}}{q_{n-1}+q_{n+1}}$$

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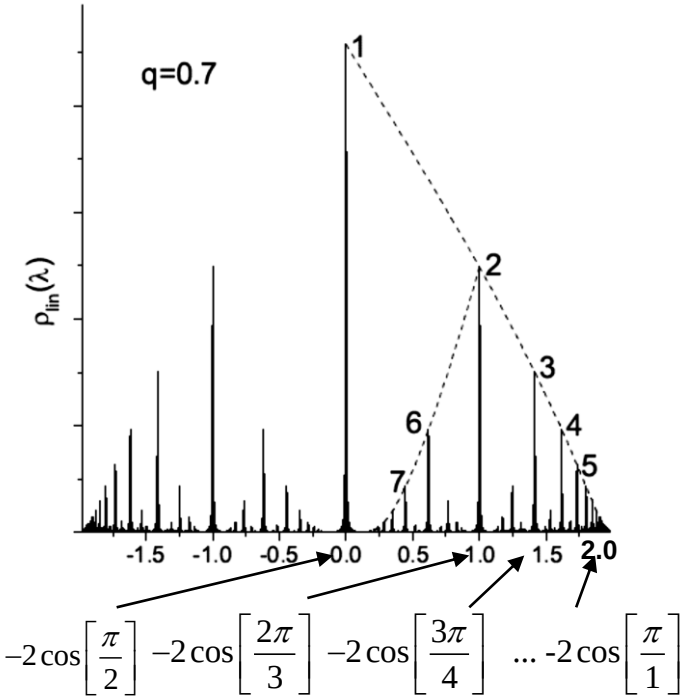
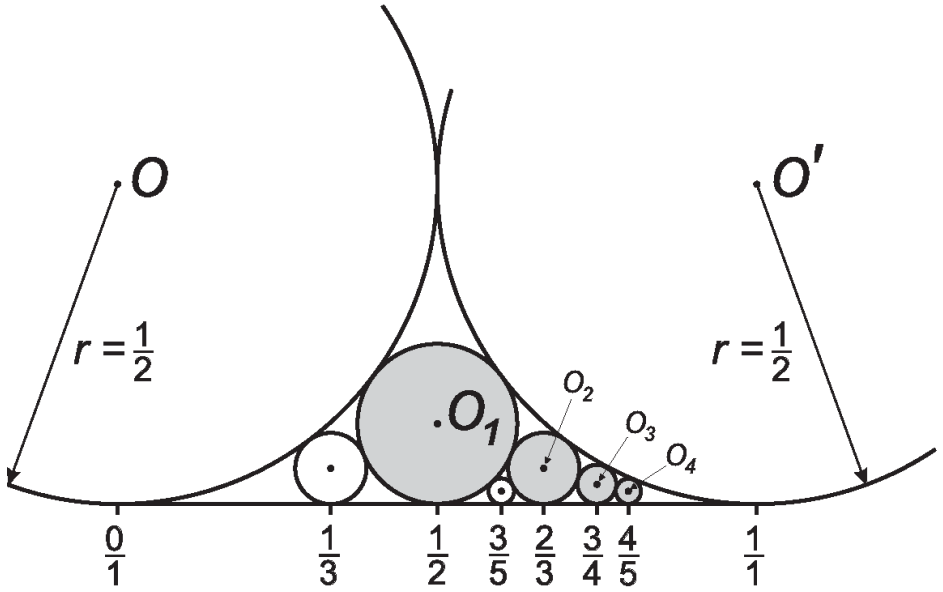
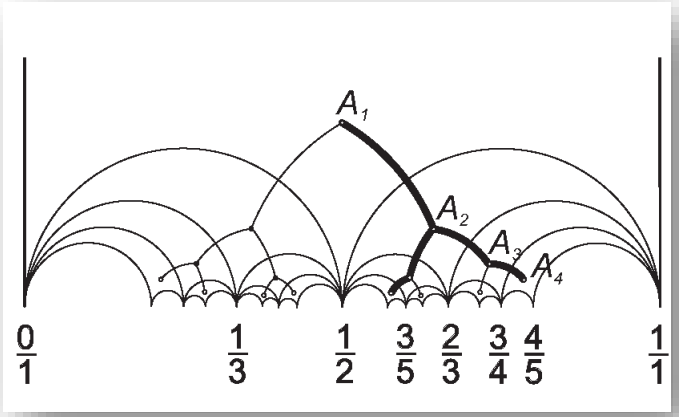


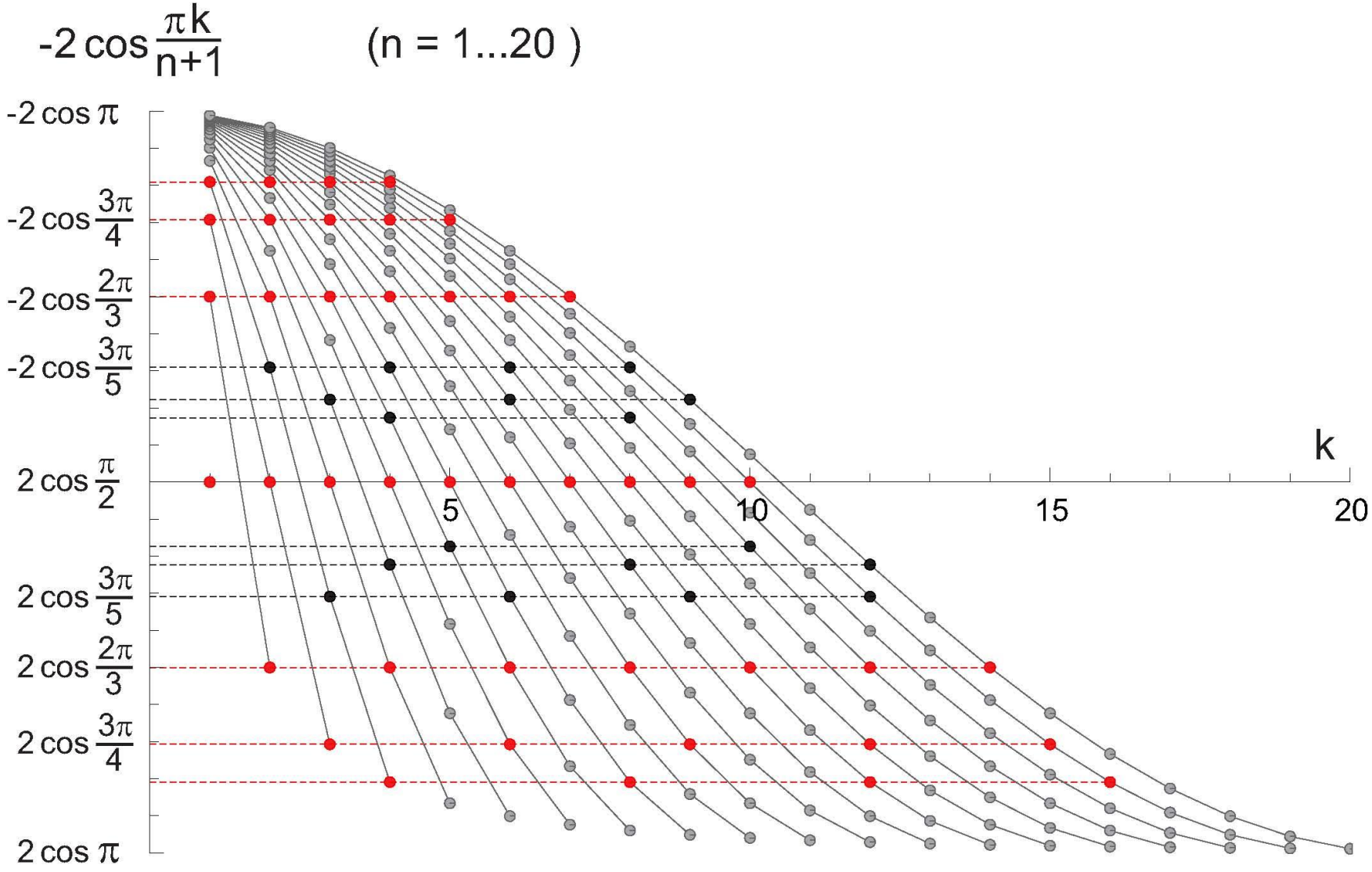


$$\frac{1}{2} \oplus \frac{1}{1} = \frac{2}{3}$$

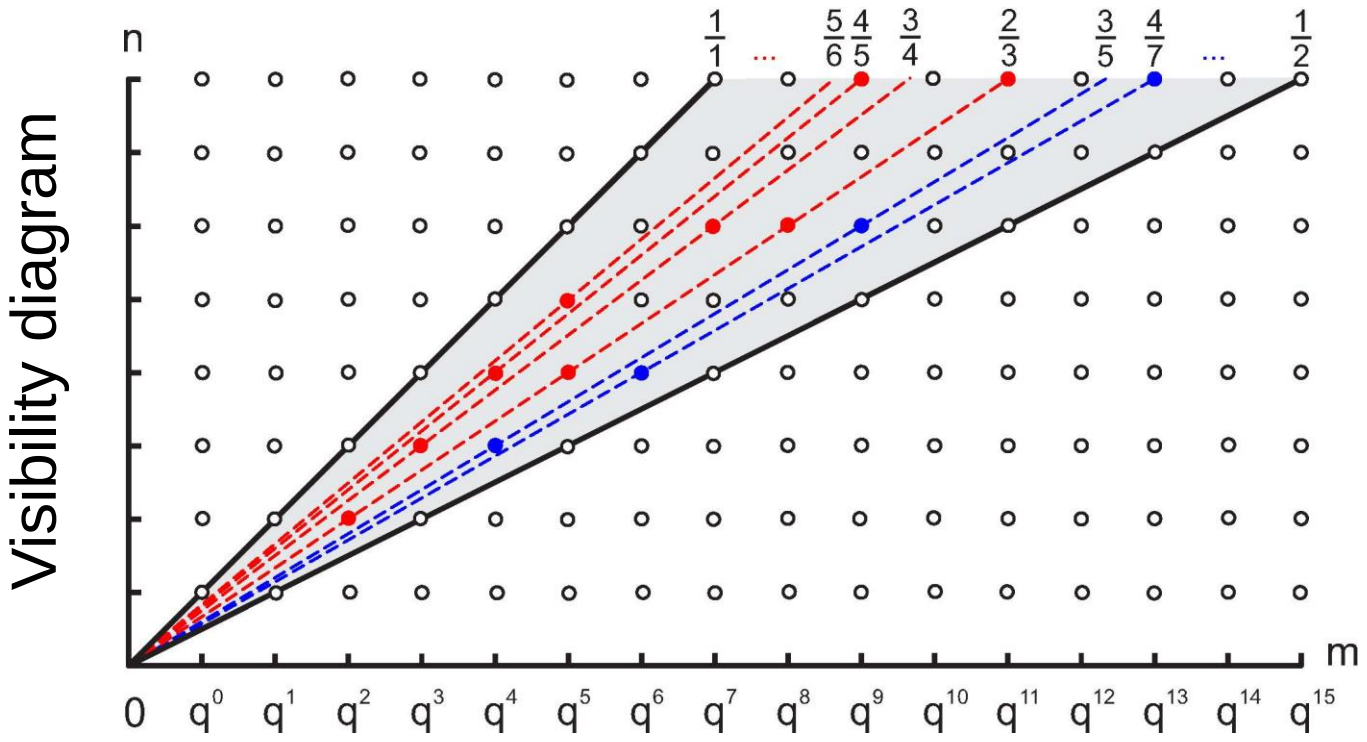


$$\frac{p_n}{q_n} = \frac{p_{n-1}}{q_{n-1}} \oplus \frac{p_{n+1}}{q_{n+1}} = \frac{p_{n-1} + p_{n+1}}{q_{n-1} + q_{n+1}}$$

Degeneracy of series $\lambda_k(n) = -2\cos\dfrac{\pi k}{n+1}$



Degeneracy of a principle series of peaks



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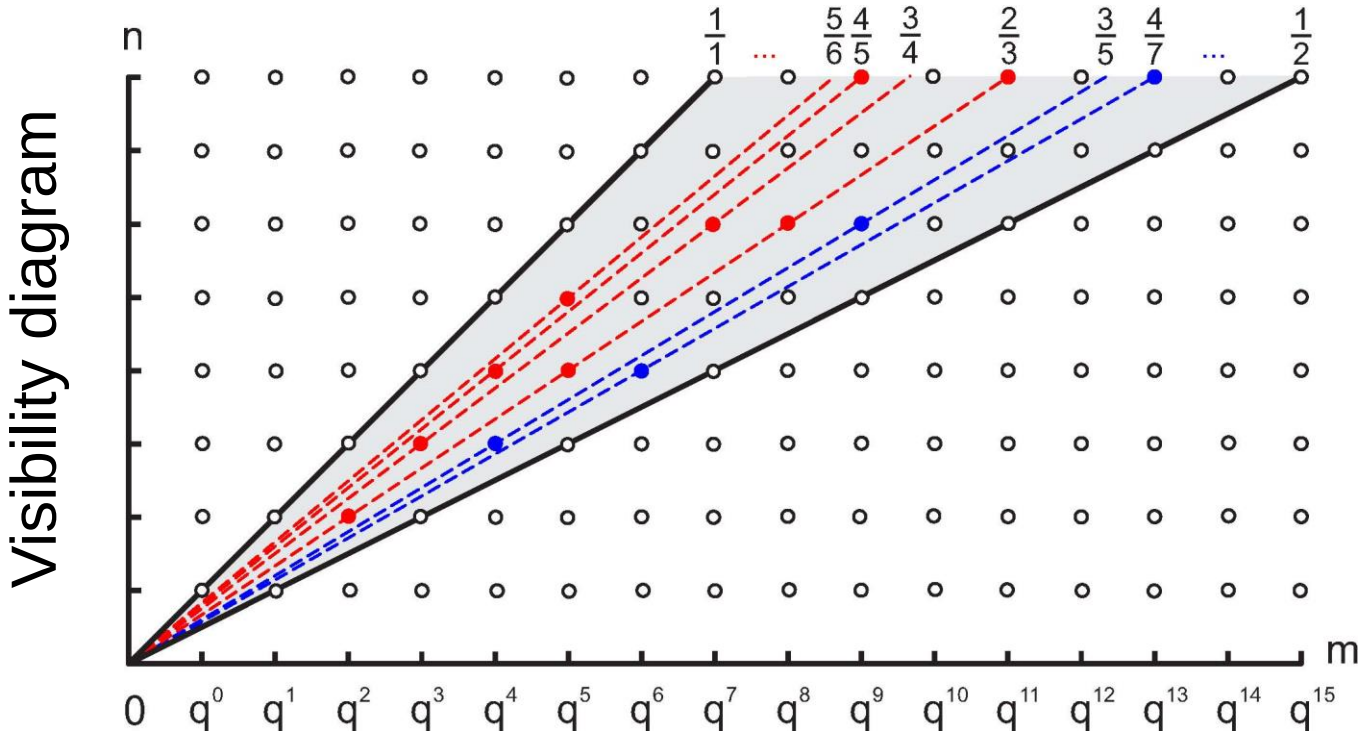
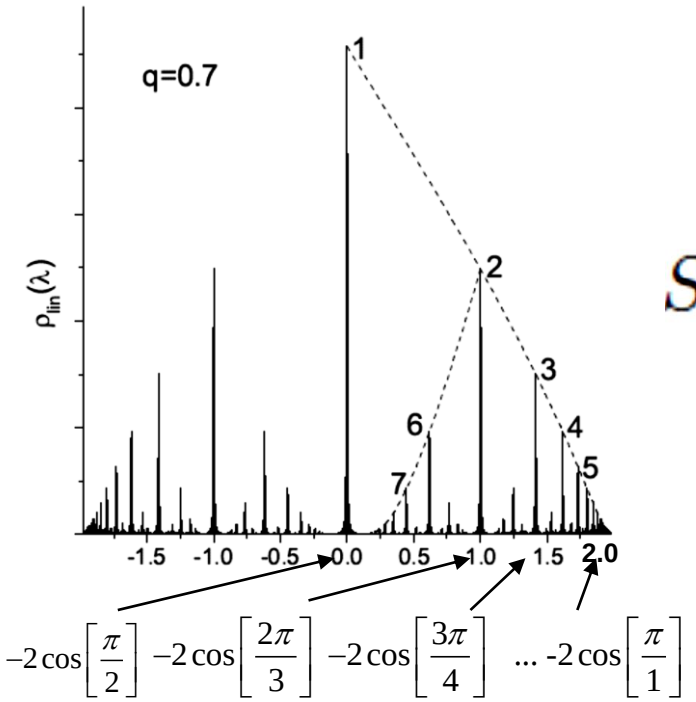


$$S_1 : \begin{cases} \lambda_1 = -2\cos\frac{\pi}{2} & \rho^{S_1}(\lambda_1) = q^1 + q^3 + q^5 + q^9 + \ldots \\ \lambda_2 = -2\cos\frac{2\pi}{3} & \rho^{S_1}(\lambda_2) = q^2 + q^5 + q^8 + q^{11} + \ldots \\ \ldots & \\ \lambda_k = -2\cos\frac{\pi k}{k+1} & \rho^{S_1}(\lambda_k) = \sum_{s=1}^{\infty} q^{(k+1)s-1} = \frac{q^k}{1-q^{k+1}} \quad (k = 1, 2, \ldots) \end{cases}$$

Spectrum tail for $q < 1$



$$S_1 : \lambda = -2\cos\frac{\pi k}{k+1}\Big|_{k\to\infty} \to 2 - \frac{\pi^2}{k^2},$$

$$\rho^{S_1}(\lambda_k)\Big|_{k\to\infty} \to q^k$$

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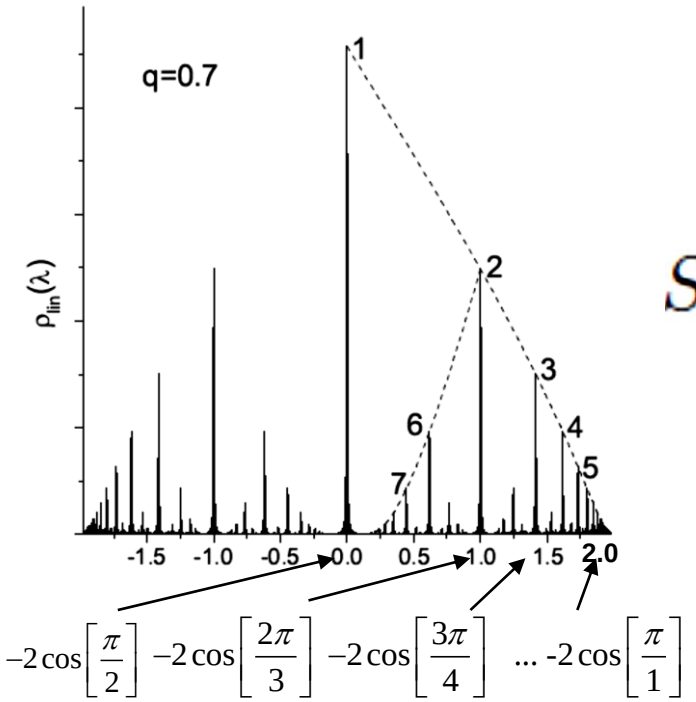

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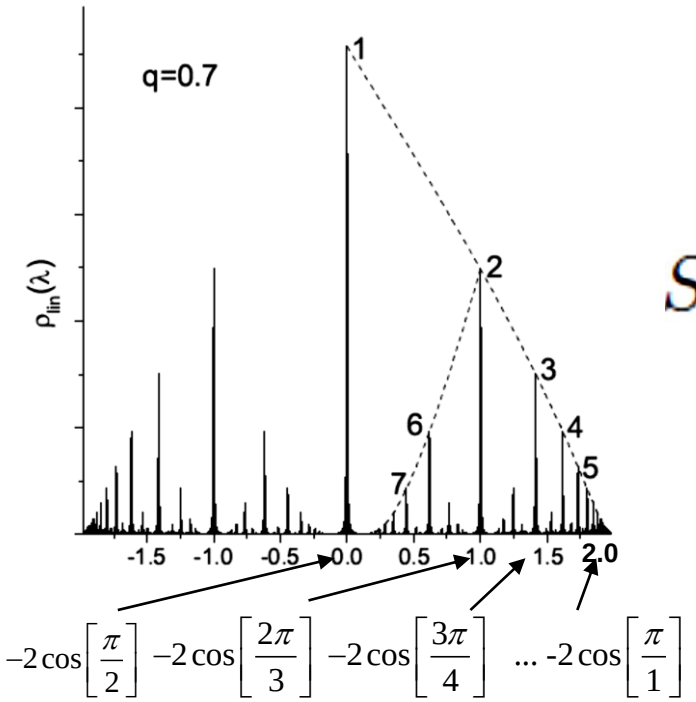
Lifshitz tail of 1D Anderson localization

$$\rho^{S_1}_{\rm lin}(\lambda \to 2) \to q^{\pi/\sqrt{2-\lambda}}.$$

Spectrum tail for $q \to 1$



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$$\rho_1^{S_1}(\lambda_k)\big|_{q\to 1} = \frac{q^k}{1-q^{k+1}}\Big|_{q\to 1}$$

$$\to \frac{1}{(k+1)(1-q)}$$

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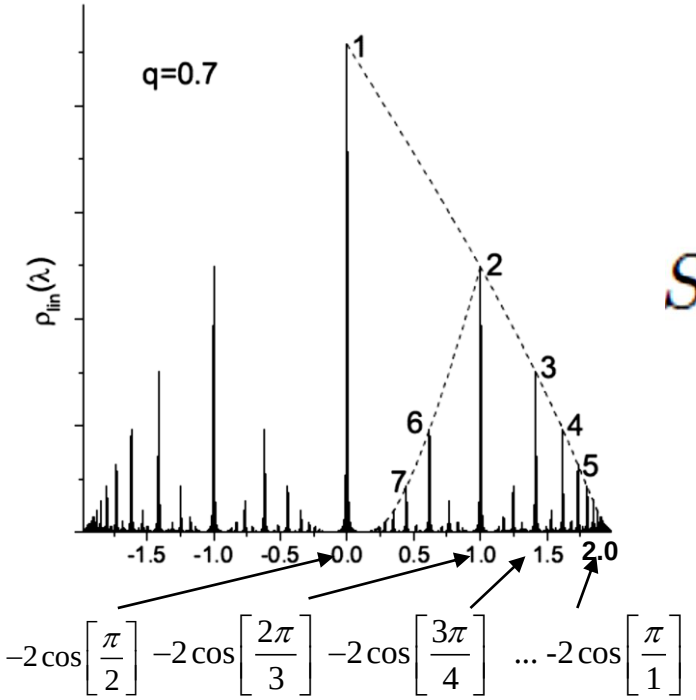

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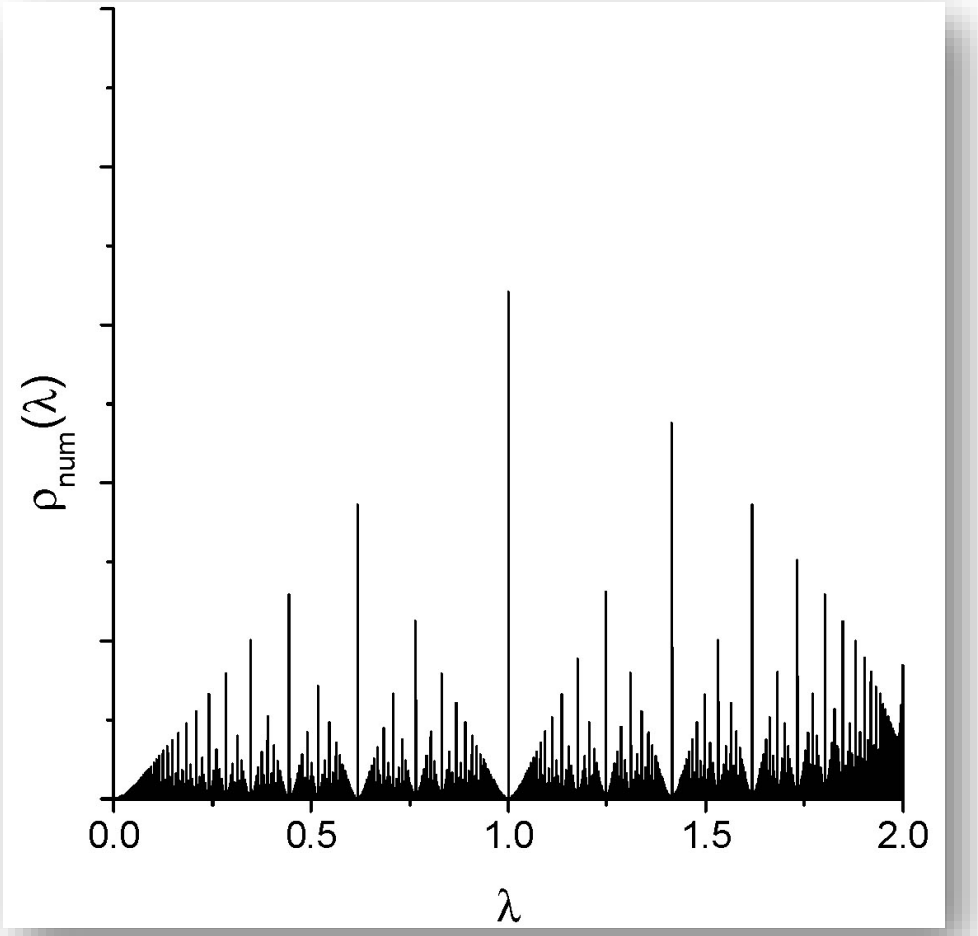
$$\rho_1^{S_1}\big|_{q\to 1}(\lambda \to 2) \to \frac{\sqrt{2-\lambda}}{\pi(1-q)}$$

Could we get analytic expression for a limiting form (at $q \to 1$) of the full spectral density, $\rho_{\text{lin}}(\lambda)$?



Spectral density $\rho_{\text{lin}}(\lambda)$ of ensemble of exponentially weighted random 3-diagonal matrices at $q \to 1$

$$\rho_{\text{lin}}(\lambda) = \lim_{\substack{N\to\infty \\ \varepsilon\to 0}} \frac{\varepsilon}{\pi N} \sum_{n=1}^{N} q^n \sum_{k=1}^{n} \frac{1}{\left(\lambda - 2\cos\frac{\pi k}{n+1}\right)^2 + \varepsilon^2}$$

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$$\lim_{q\to 1^-} \frac{\rho_{\mathrm{lin}}(\lambda, q)}{\sqrt{-\ln\left|\eta\left(\frac{1}{\pi}\arccos(-\lambda/2) + i\frac{(1-q)^2}{12\pi}\right)\right|}} = 1$$

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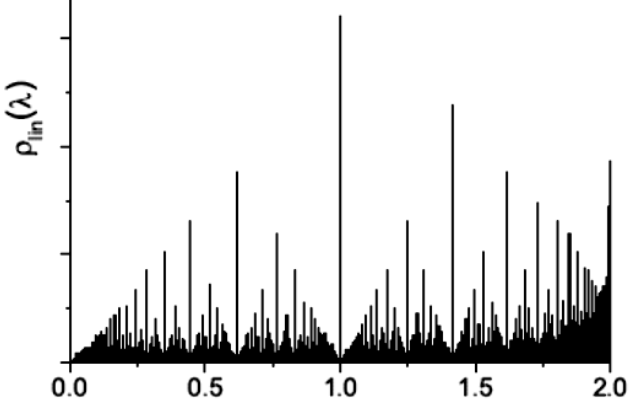
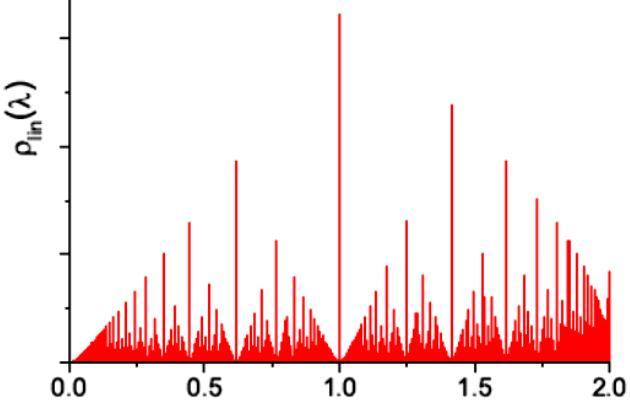
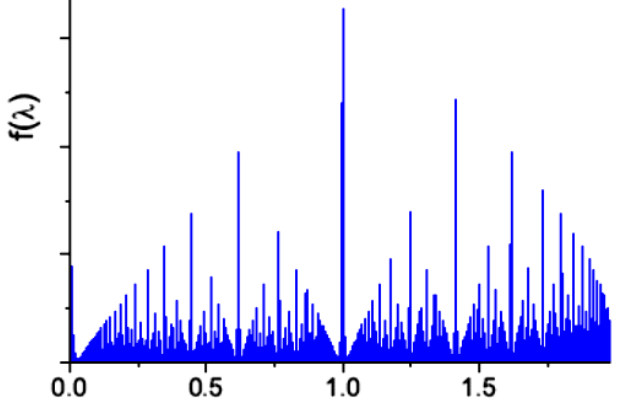
$\rho_{\text{lin}}(\lambda)$ computed via Monte-Carlo



Numeric summation of the series for $\rho_{\text{lin}}(\lambda)$



Analytic expression of $\rho_{\text{lin}}(\lambda)$ via Dedekind η



Modular Dedekind η – function is invariant with respect to SL(2,Z) group

$$\eta(z) = e^{\pi i z/12} \prod_{n=0}^{\infty} (1 - e^{2\pi i n z}); \quad z = x + iy \quad (y > 0)$$

$$\eta(z+1) = e^{\pi i z/12}\, \eta(z)$$

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Define $f(z) = \text{const} \,|\, \eta(x+iy) \,|\, y^{1/4}$

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$f(z)$ obeys duality relation

$$f\left(\frac{k}{k+1} + iy\right) = f\left(\frac{1}{k+1} + \frac{i}{(k+1)^2 y}\right)$$

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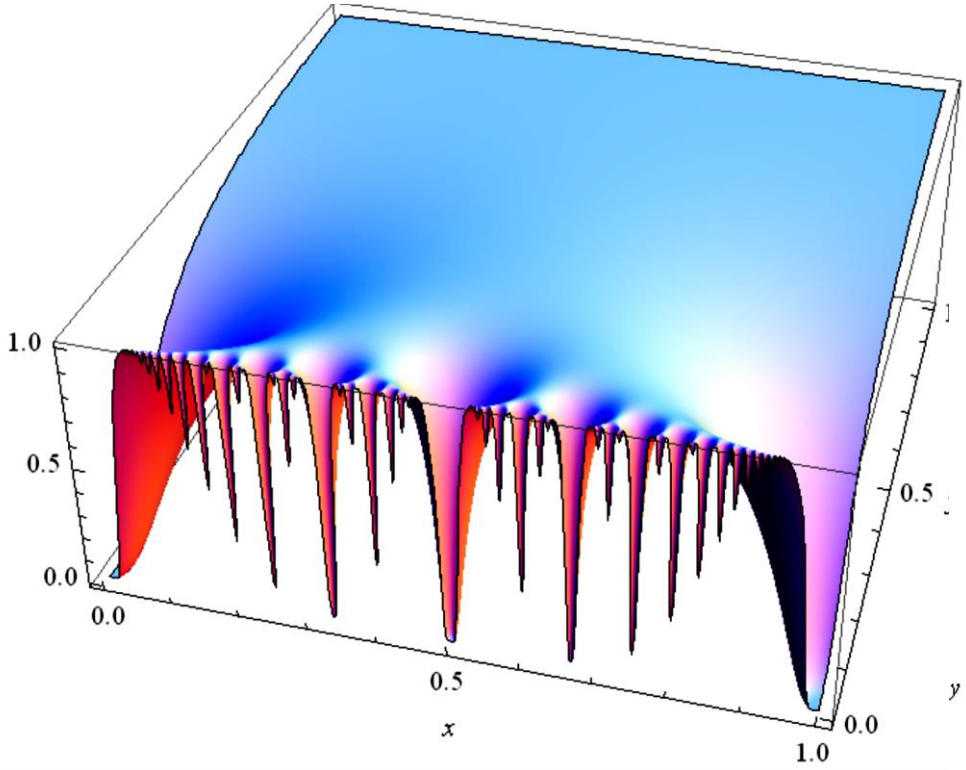

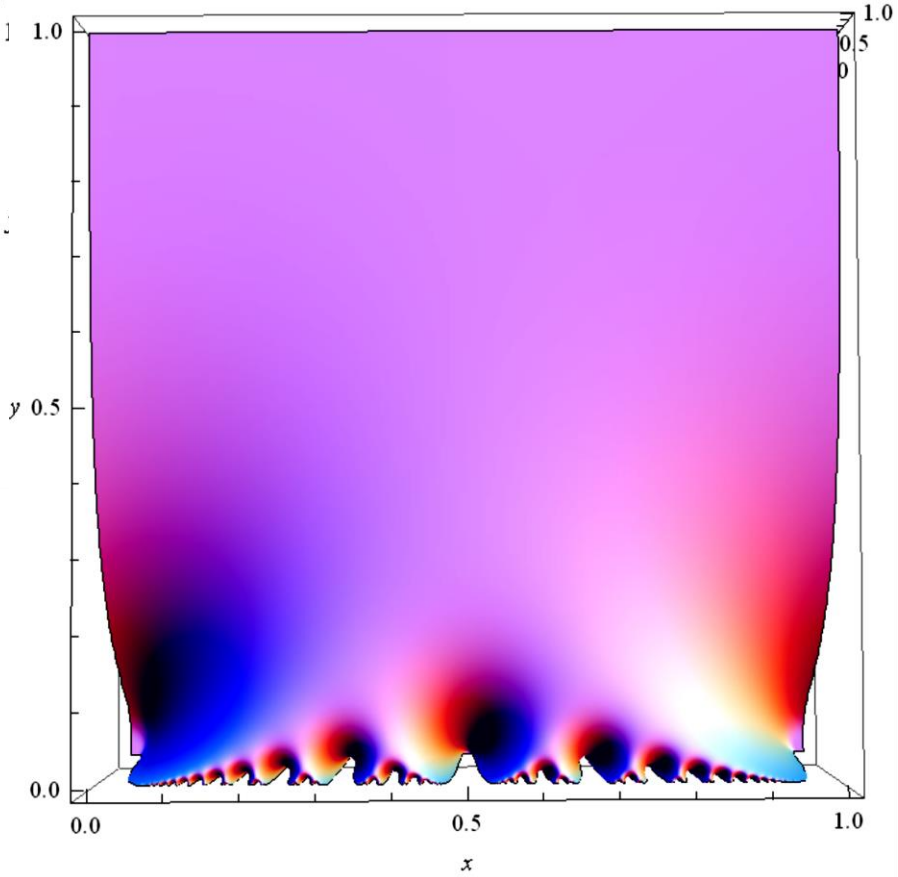

Some plots on the basis of Dedekind η – function

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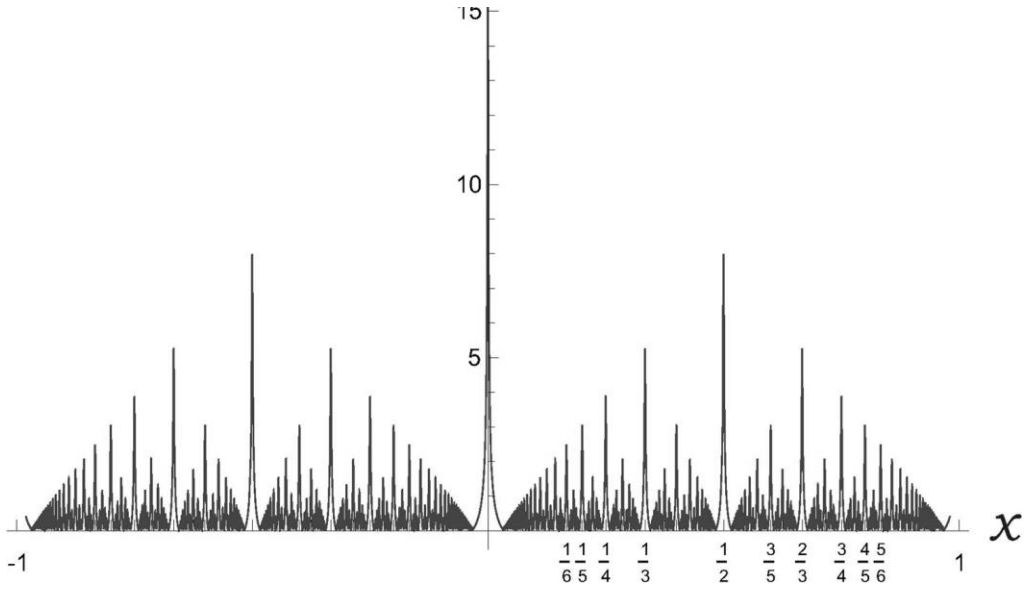

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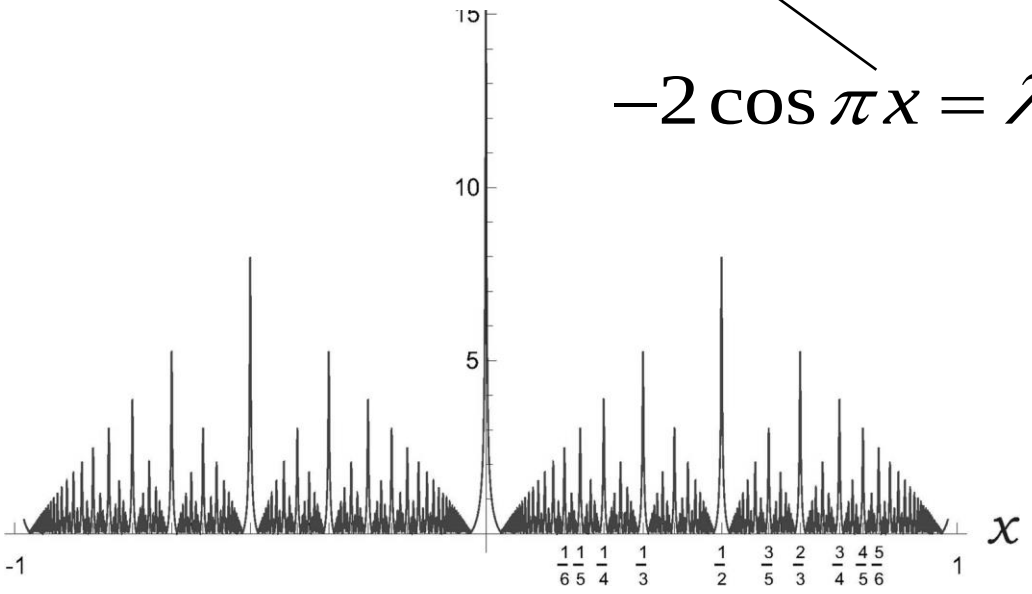

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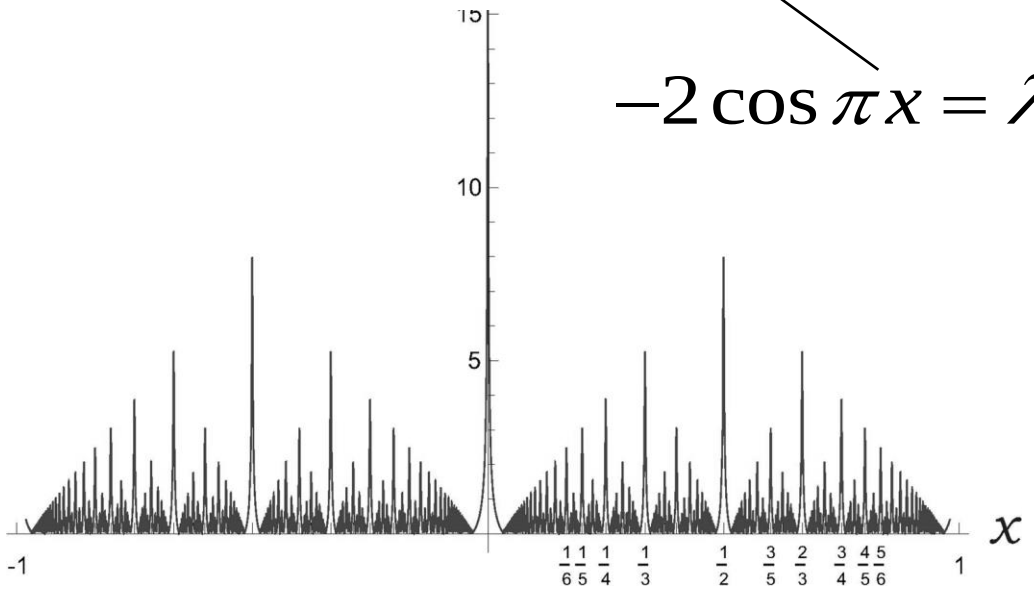
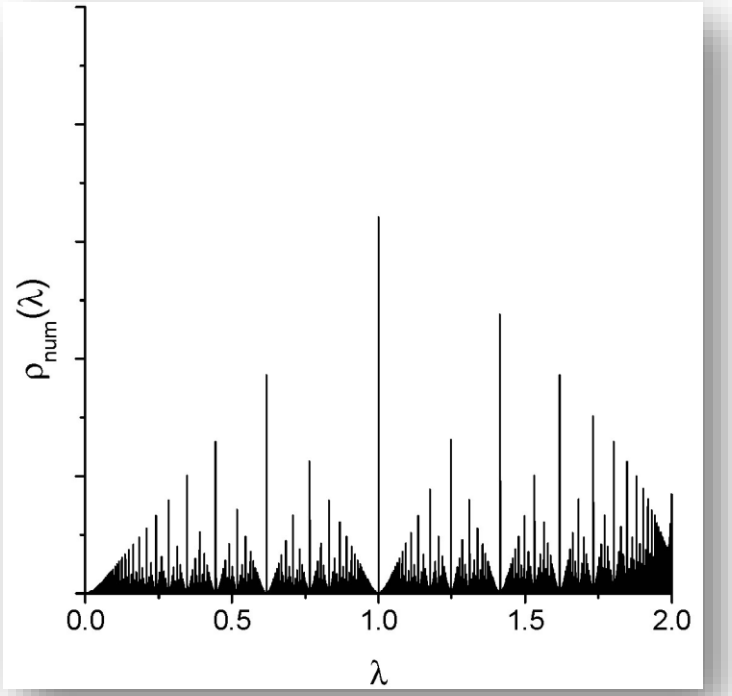
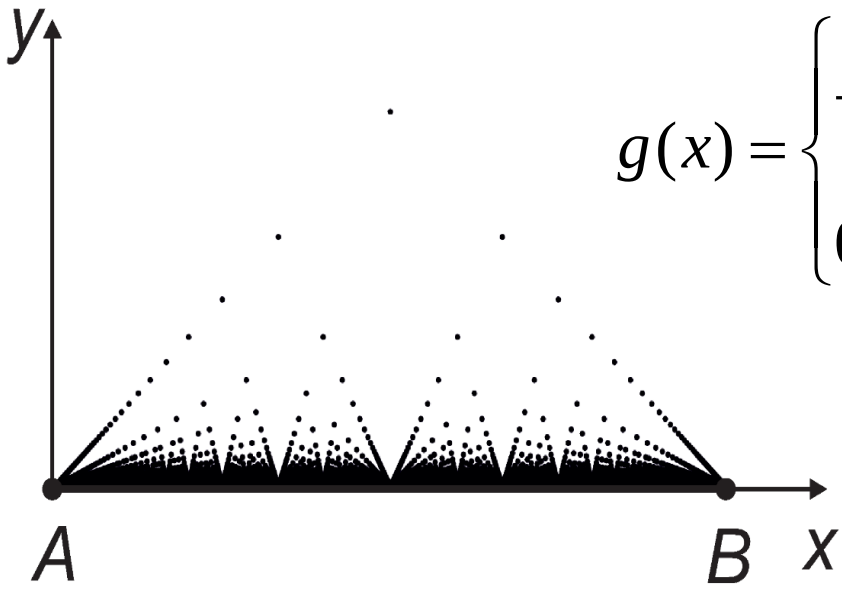




Thomae function, Dedekind η and Euclid orchard



$$g(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ and } (p, q) \text{ are coprime} \\ 0 & x \text{ otherwise} \end{cases}$$

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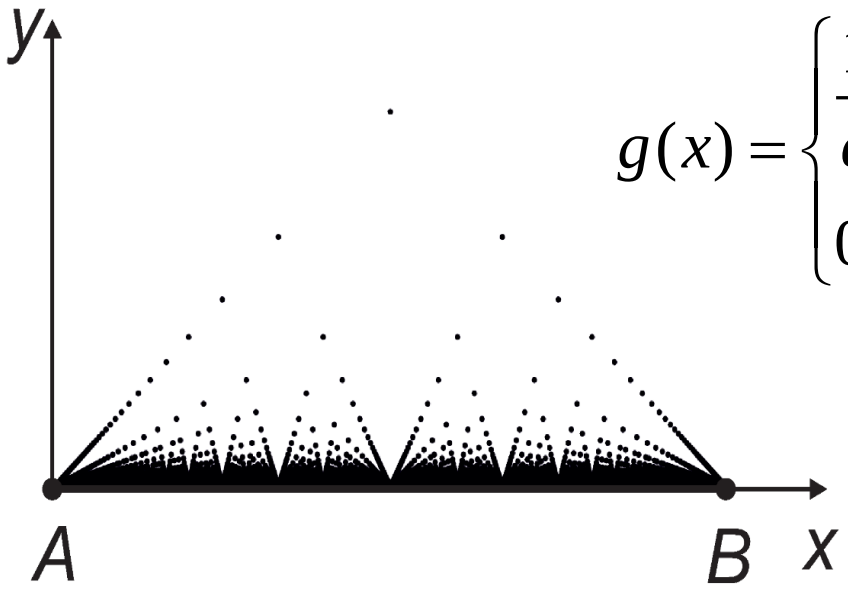

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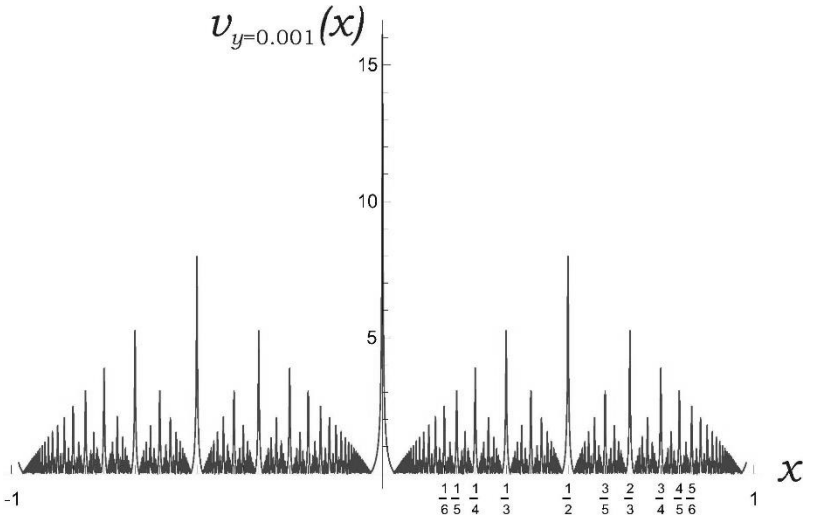

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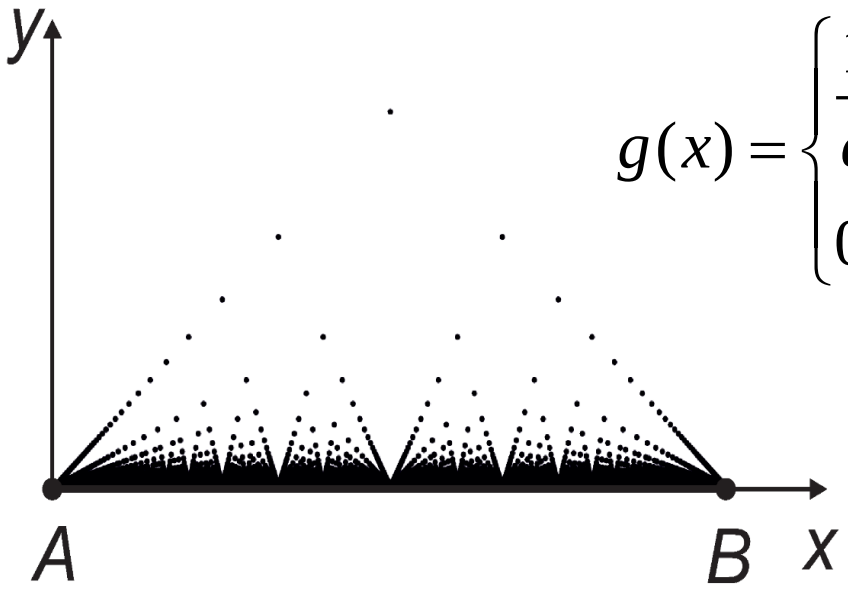
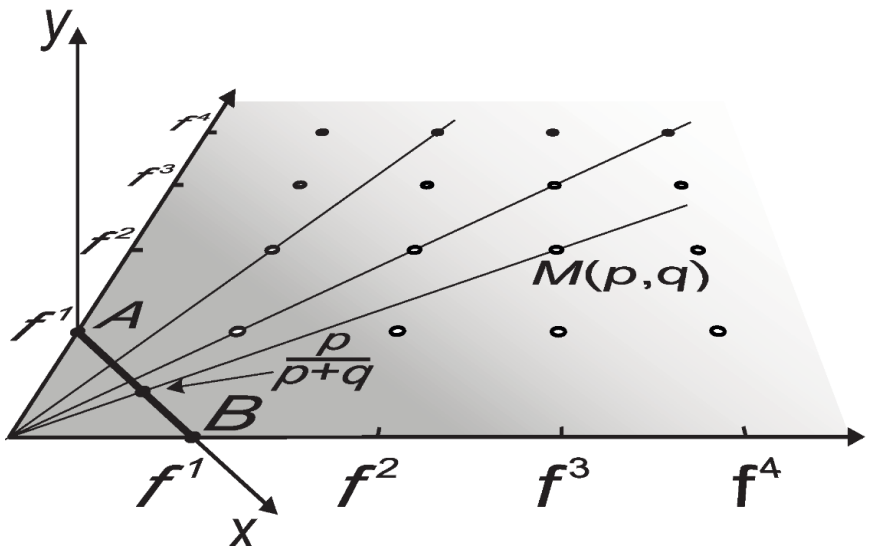
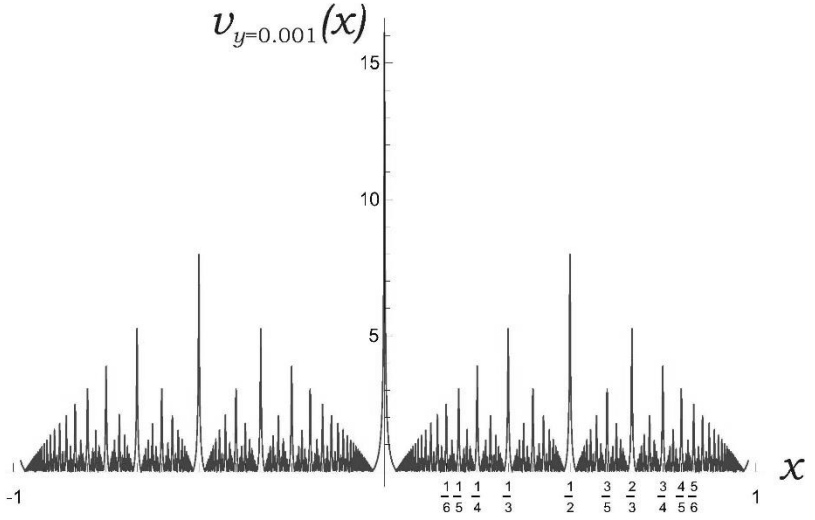
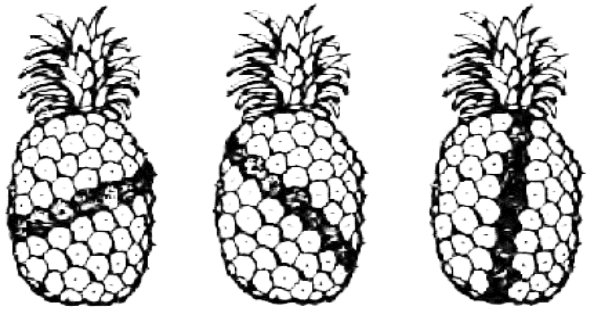


$$g(x)=\begin{cases}\frac{1}{q} & \text{if } x=\frac{p}{q} \text{ and } (p,q) \text{ are coprime}\\ 0 & x \text{ otherwise}\end{cases}$$

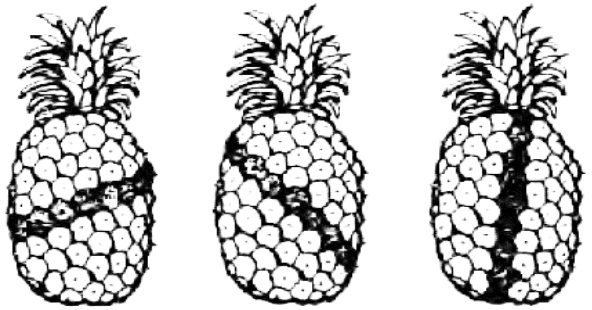




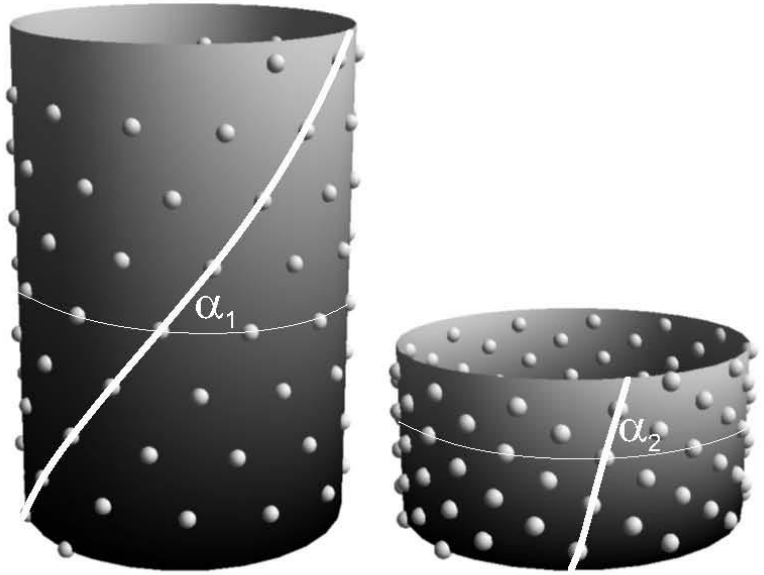
Phyllotaxis



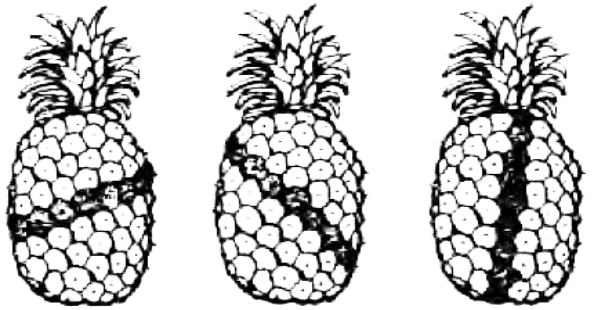
Phyllotaxis



Energetic approach to phyllotaxis, L. Levitov, (1991)

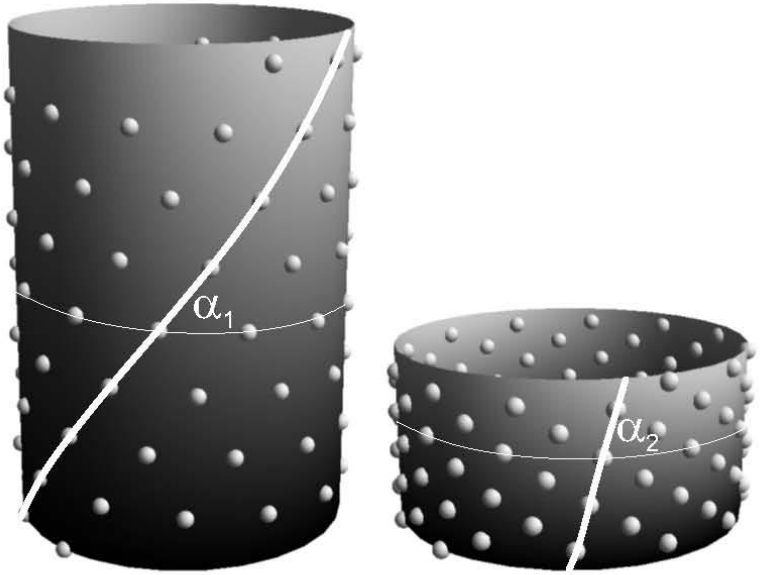


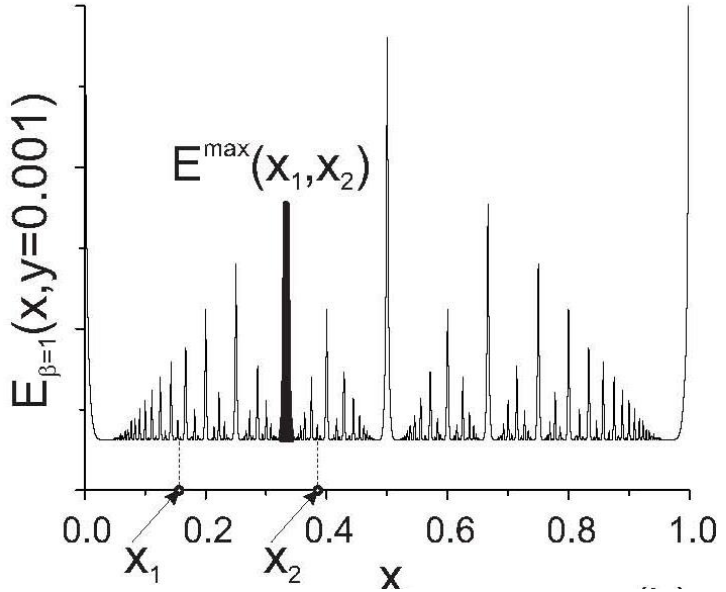
Phyllotaxis



Energetic approach to phyllotaxis, L. Levitov, (1991)

$$x = \frac{\alpha}{2\pi},\ y = \frac{h}{2\pi} \qquad r_{n,m} = \left(\frac{m+nx}{\sqrt{y}}, n\sqrt{y}\right) \qquad E = \sum_{n,m} e^{-\frac{c(m+nx)^2}{y} - cyn^2}$$

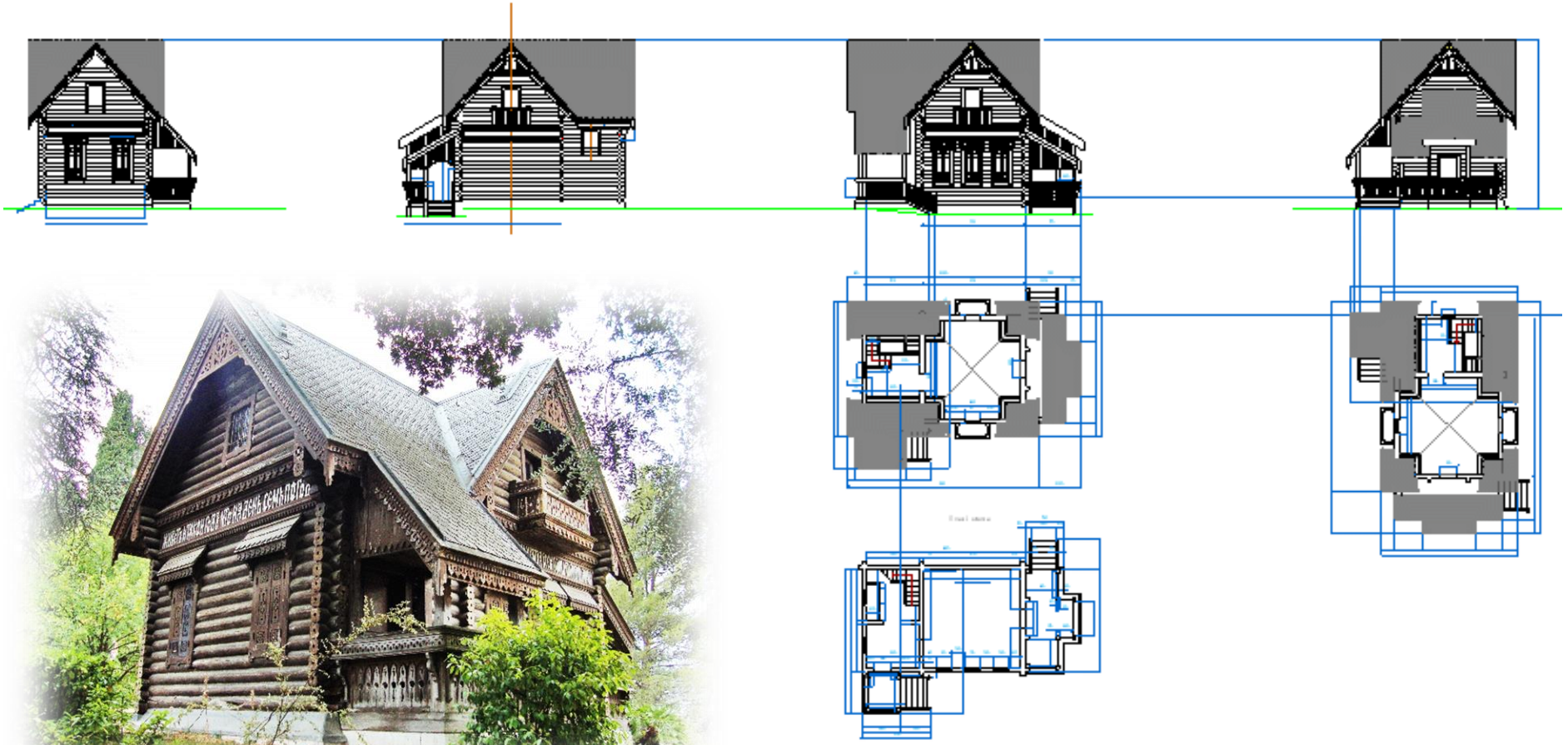




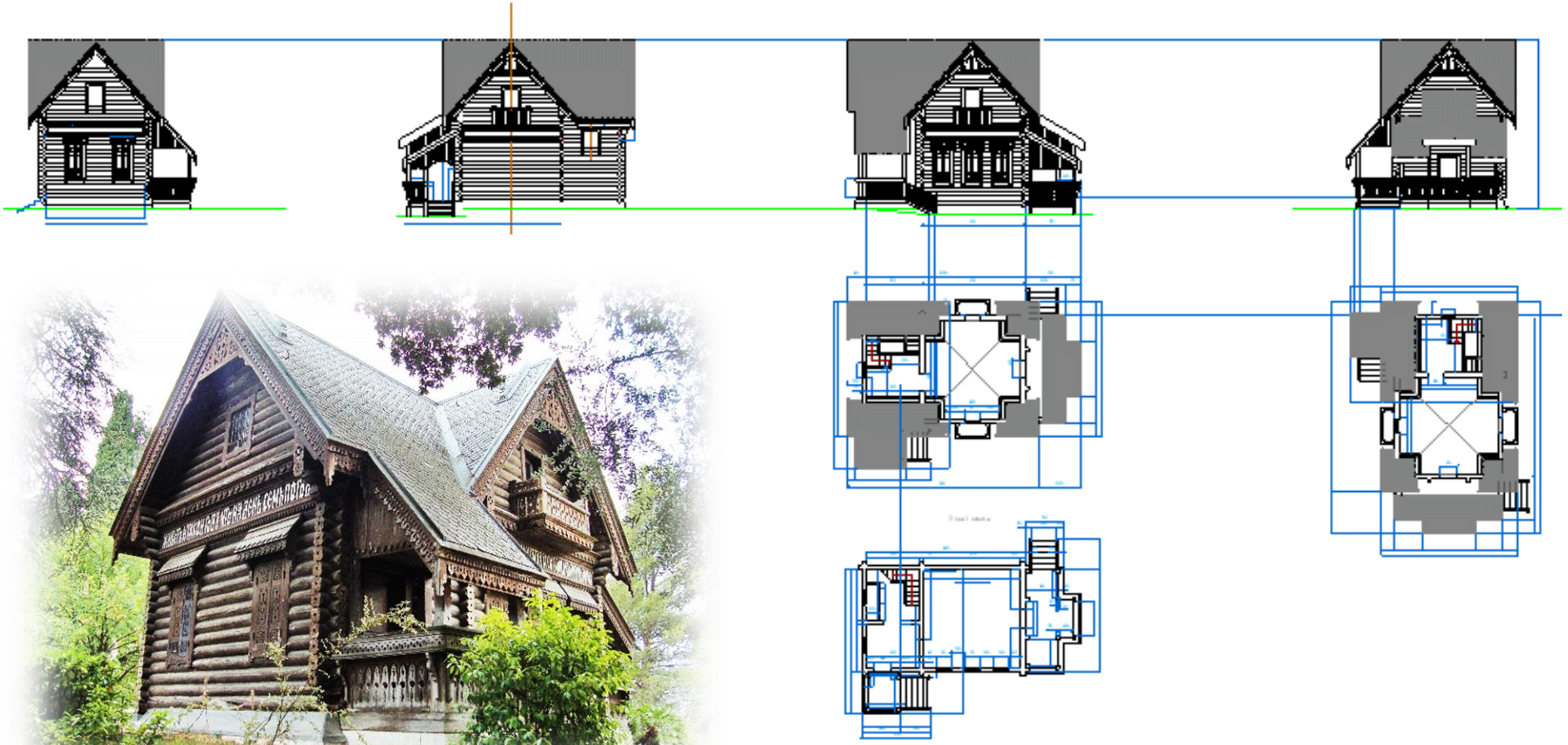
Hypothesis

- Hierarchical organization occurs in collective variables when conformational space is huge, and statistics is rare.

- Such a situation is natural for protein folding, analysis of statistical properties of genome, "large data", etc ...



Measurements: ITMO (St Petersburg)



Measurements: ITMO (St Petersburg)

Is IZBA a number-theoretic fractal…?