Groups of diffeomorphisms of Cantor sets

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June 12, 2018

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$\mathrm{Homeo}(S)$ can be replaced by $\mathrm{Diff}(S)$.
Much has been said in the case where S is a circle. Here the space S we will act on is a Cantor set.

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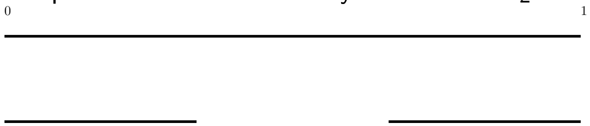

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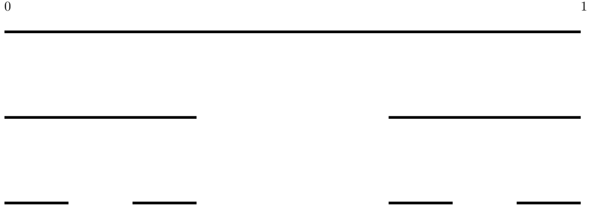

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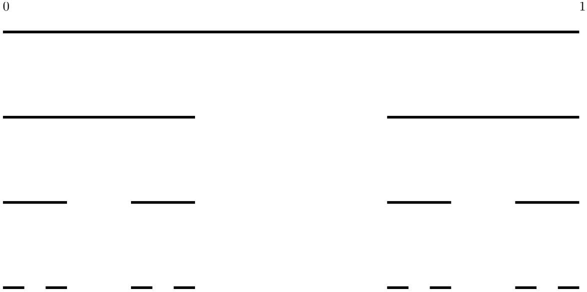

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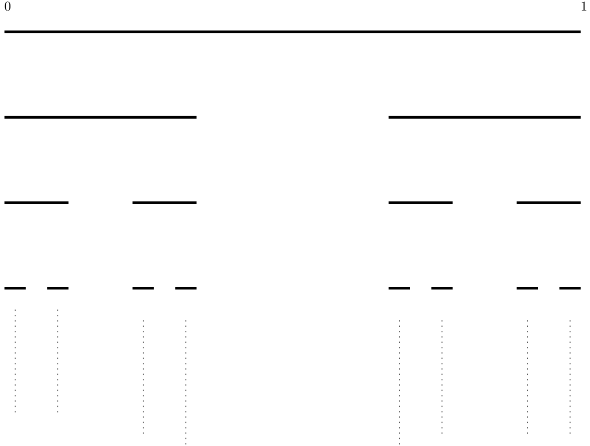

We call structural interval of K_2 any set of the form $I \cap K_2$, where I is one of the intervals appearing in the construction of K_2.

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If, at each step, we decide to remove $n-1$ intervals of our segment (instead of one), we obtain a Cantor set which we call K_n.

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$$\mathfrak{diff}^r(K) = \left\{ f_{|K},\ f \in \mathrm{Diff}^r(M),\ f(K) = K \right\}.$$

This group is independent of the embedding $\mathbb{R} \subset M$ and of the chosen surface (or manifold) M.

Let $K \subset \mathbb{R}$ be a Cantor set.

Definition (An equivalent definition)

We call C^r diffeomorphisms of K any homeomorphism f of K such that, for any point x of K, there exists an open interval I of $\mathbb{R}$ and a diffeomorphism $\tilde{f} : I \to \tilde{f}(I)$ such that $f_{|I \cap K} = \tilde{f}_{I \cap K}$.

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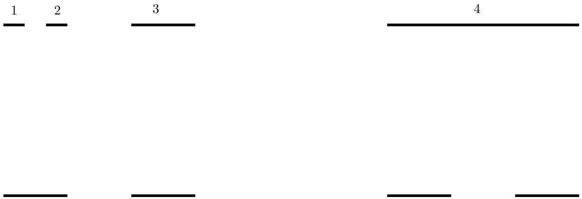

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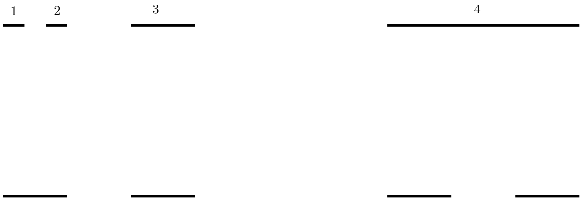

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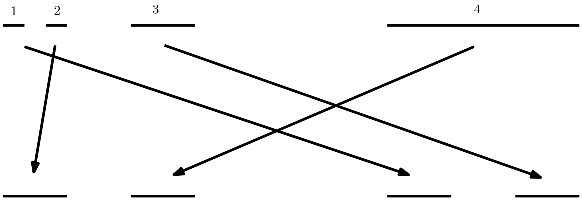

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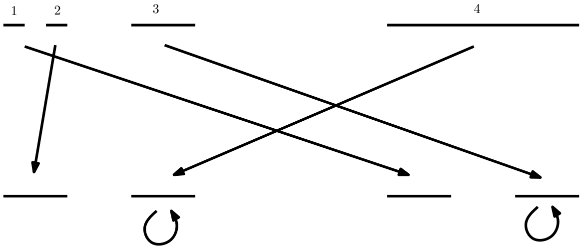
2. Choose a bijection between those two partitions: this gives a diffeomorphism of K_2.
3. Choose intervals of the second partition and flip them (make a symmetry with respect to the center of the interval).

After those three steps, we obtain a C^∞-diffeomorphism of K_2.



Theorem (Funar-Neretin)

The group $\mathfrak{Diff}^1(K_2) = \mathfrak{Diff}^\infty(K_2)$ *is the group consisting of diffeomorphisms obtained by this procedure.*

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The group consisting of elements obtained after the first two steps of the procedure is the Higman-Thompson group V_2.
If we replace K_2 by K_n, with $n \geq 3$, we obtain similar results and the Higman-Thompson group V_n is a subgroup of $\mathfrak{Diff}^\infty(K_n)$.

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Schur proved in 1911 that any finitely generated periodic subgroup of $\mathrm{GL}_n(\mathbb{R})$ is finite.
Much later (1964), Golod and Shafarevich found a counterexample and, nowadays, multiple counterexamples to this conjecture have been found.

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Taking the particular cases of the Cantor sets K_n, this provides a new proof of the following theorem.

Theorem (Rover)

Any finitely generated periodic subgroup of V_n is finite.

Tools : hyperbolic dynamics (Sacksteder theorem) and the Thurston stability theorem.

Theorem (Malicet-Militon)

Let G be a finitely generated nilpotent subgroup of $\mathfrak{Diff}^2(K)$ without torsion elements. Then G is virtually abelian.

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Hope : prove the same thing for any finitely generated subgroup of $\mathfrak{Diff}^2(K)$ without free semi-groups on two generators.