Gluing bifurcations for monotone families of vector fields on a torus

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Gluing bifurcations

C. R. Acad. Sc. Paris, t. 299, Série I, n° 7, 1984 253

SYSTÈMES DYNAMIQUES. — Une nouvelle bifurcation de codimension 2 : le collage de cycles.

Note de **Pierre Coullet, Jean-Marc Gambaudo** et **Charles Tresser,** présentée par René Thom.

Remise le 14 mai 1984.

Nous décrivons une nouvelle bifurcation de codimension 2 pour des champs de vecteurs dans $\mathbb{R}^n$, $n \geqq 3$.

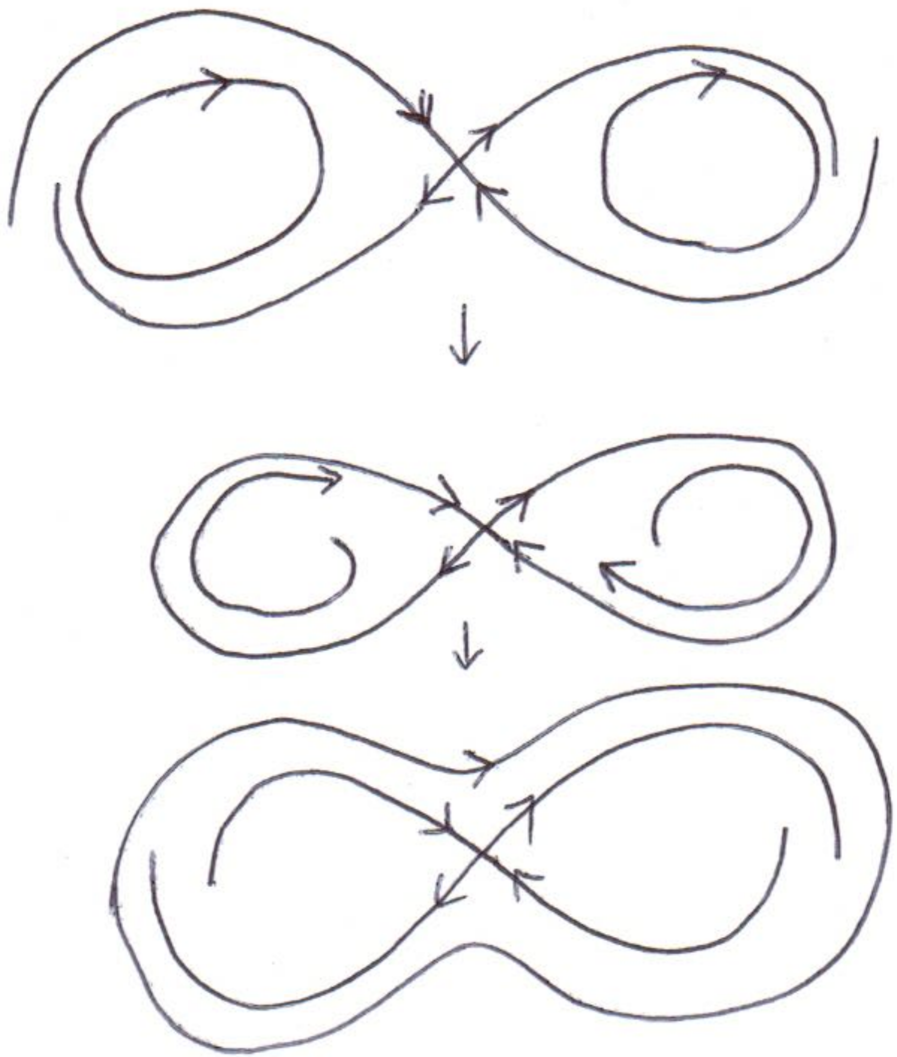
DYNAMICAL SYSTEMS. — A New Bifurcation of Codimension Two: the Gluing of Cycles.

We describe a new bifurcation of codimension two for vector fields in $\mathbb{R}^n$, $n \geqq 3$.

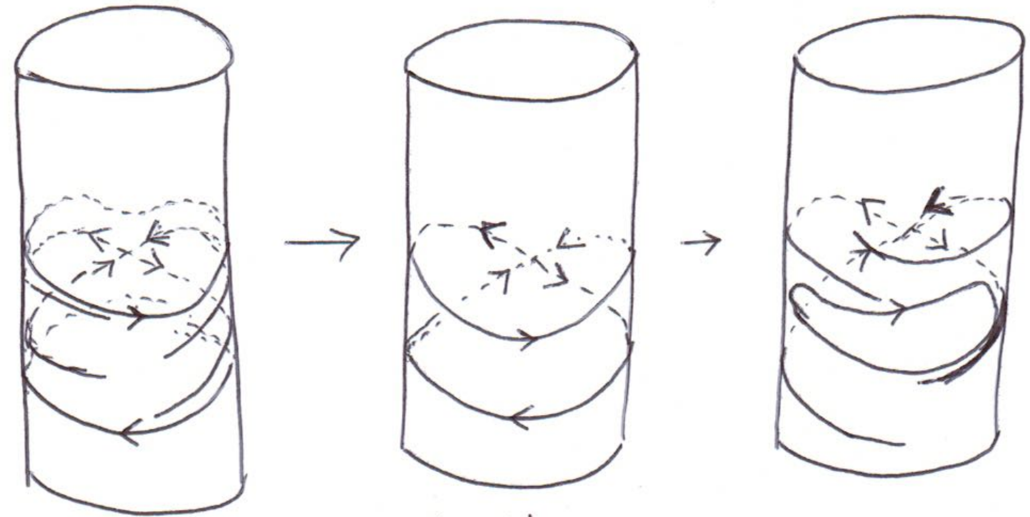
(and Glendinning, 1984, for $\mathbb{Z}_2$-symmetric case).

But already interesting in 2D (Turaev, 1984).

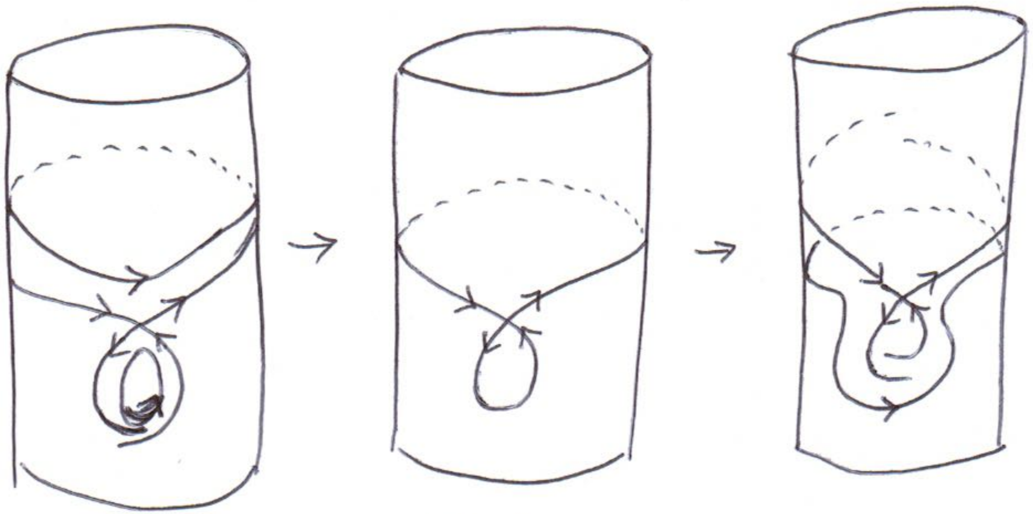
Gluing with $\mathbb{Z}_2$-symmetry in plane



Homotopically non-trivial versions on cylinder [BGKM91]

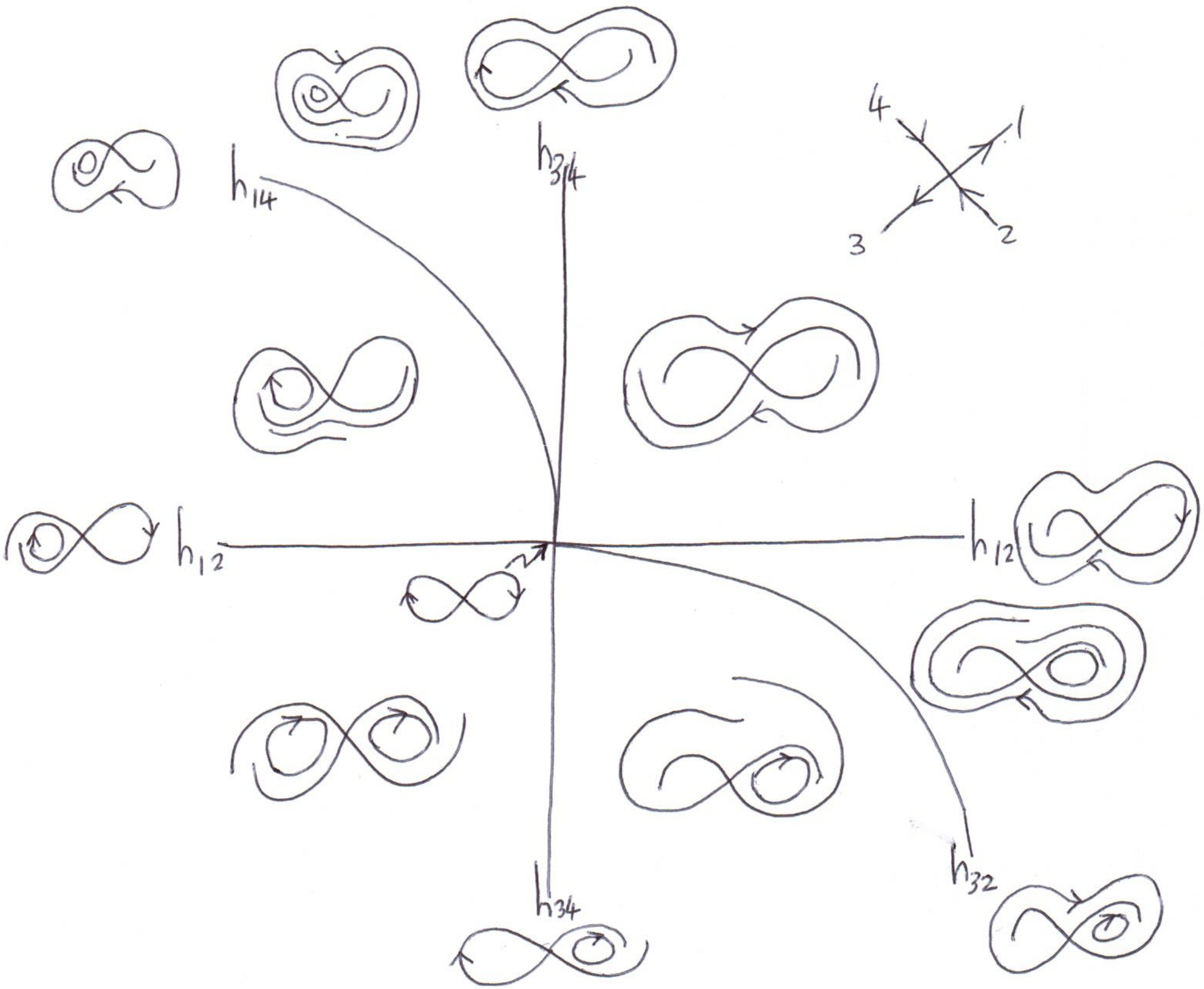


Necklace



Pendant

Full bifurcation diagram has two parameters



Monotone families of vector fields on the torus

$$\dot{x} = G(\Omega, x), \quad \Omega \in \mathbb{R}^2, \quad x \in \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2, \quad G \in C^3[\mathbb{R}^2 \times \mathbb{T}^2],$$

with $c > 0$ such that $\langle d_\Omega G\, \omega, \omega \rangle \geq c\, |\omega|^2$ for all tangent vectors ω to $\mathbb{R}^2$.

Example:

$$G : \begin{cases} \dot{x} = \Omega_x - \cos 2\pi y - \varepsilon \cos 2\pi x \\ \dot{y} = \Omega_y - \sin 2\pi y - \varepsilon \sin 2\pi x \end{cases}$$

Assume all bifurcations are codimension-one or two and the family is transverse to them.

Study the simplest cases, meaning that the numbers of a sequence of types of object are minimised.

We nevertheless obtain quite complicated bifurcation diagrams, including lots of gluing bifurcations.

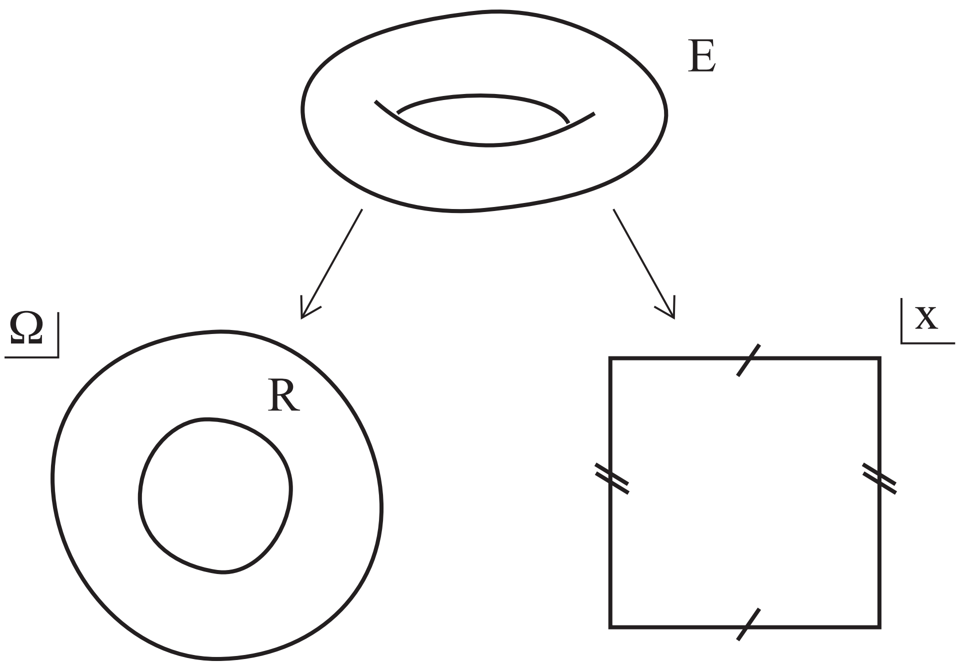
1. Equilibria: By the monotonicity assumption the set

$$E := \{(\mathbf{\Omega}, \mathbf{x}) \in \mathbb{R}^2 \times \mathbb{T}^2 : G(\mathbf{\Omega}, \mathbf{x}) = 0\}$$

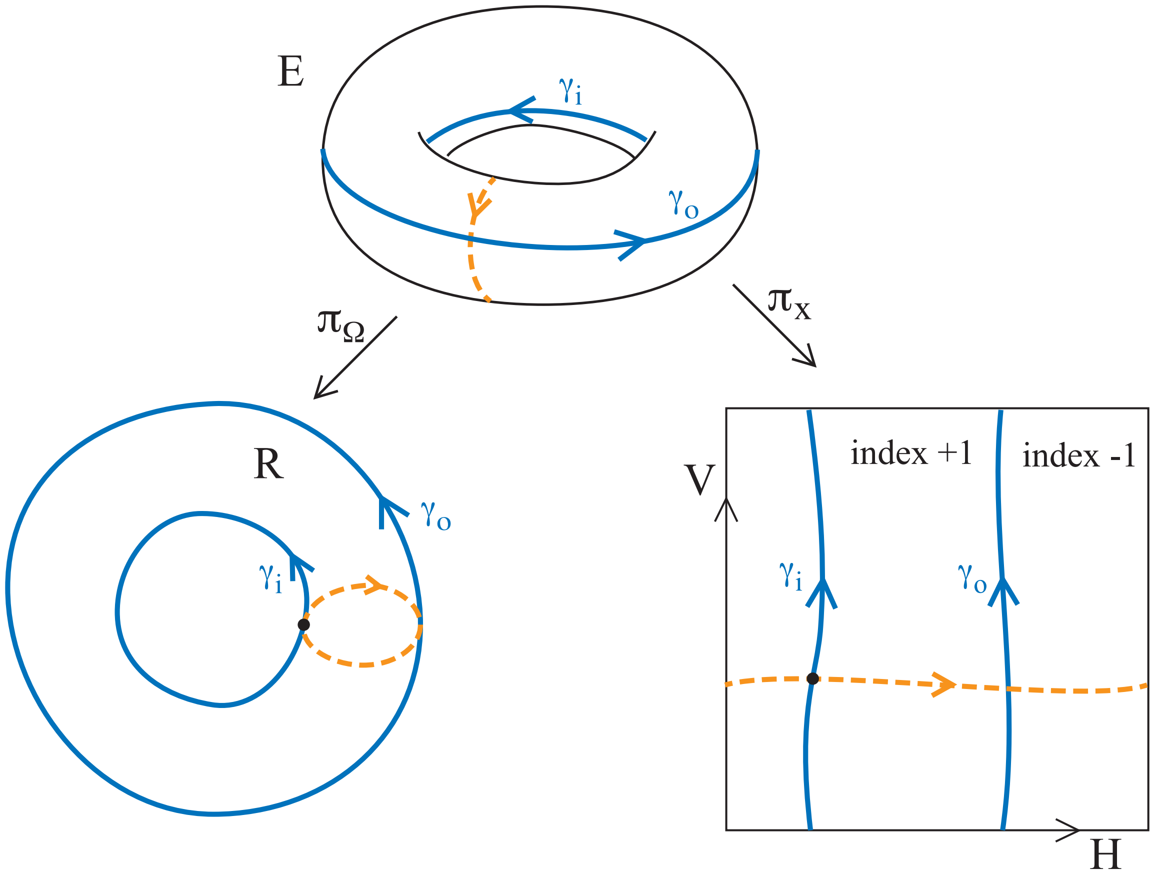
of equilibria is a graph over $\mathbb{T}^2$: $\mathbf{\Omega} = \mathbf{\Omega}_E(x)$.

Its projection R to $\mathbb{R}^2$ has saddle-node boundaries, possibly with cusps and self-intersections.

- Assume **at most two equilibria.** Then R is an **annulus**.



Principal homotopy classes: Going round a saddle-node equilibrium boundary γ (sne) anticlockwise defines a non-trivial homotopy class on E, which we call *vertical*.
Going around E to cross the two fold curves without making a revolution around R defines another homotopy class, which we call *horizontal*.
The equilibria have index ± 1 as shown.



2. Bogdanov-Takens points (B pts):

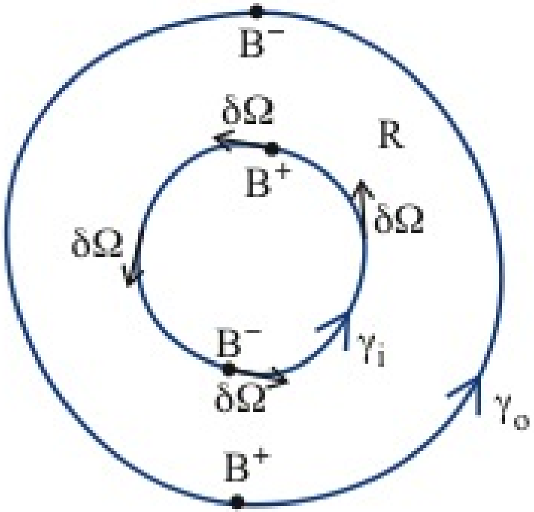
Follow a sne curve γ for one vertical revolution. Its tangent vector $(\delta\Omega, \delta x)$ has $\begin{cases} \delta\Omega \neq 0 \text{ and makes one revolution} \\ \delta x \neq 0 \text{ and makes no revolutions} \end{cases}$.

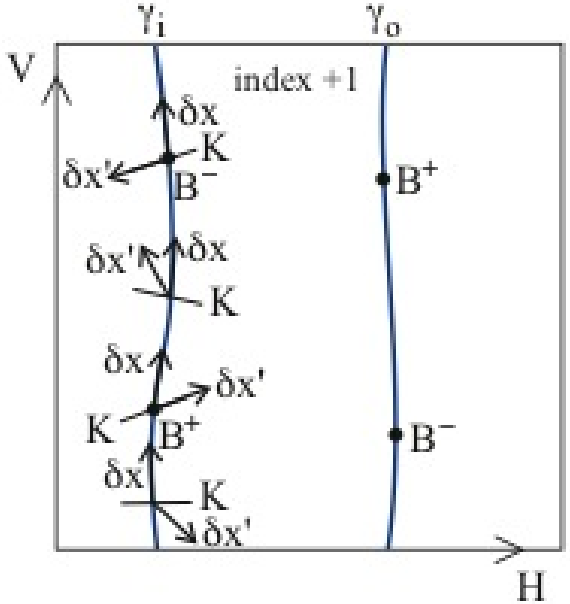
$G(\Omega, x) = 0$ implies the components are related by

$$-d_\Omega G\,\delta\Omega = +d_x G\,\delta x =: \delta x'.$$

By monotonicity, $d_\Omega G$ is invertible so $\delta x'$ makes one revolution.

$K := \ker d_x G$ is 1D and transverse to δx, so makes no revolutions. Thus $\delta x'$ lies in K at least twice, giving B points.





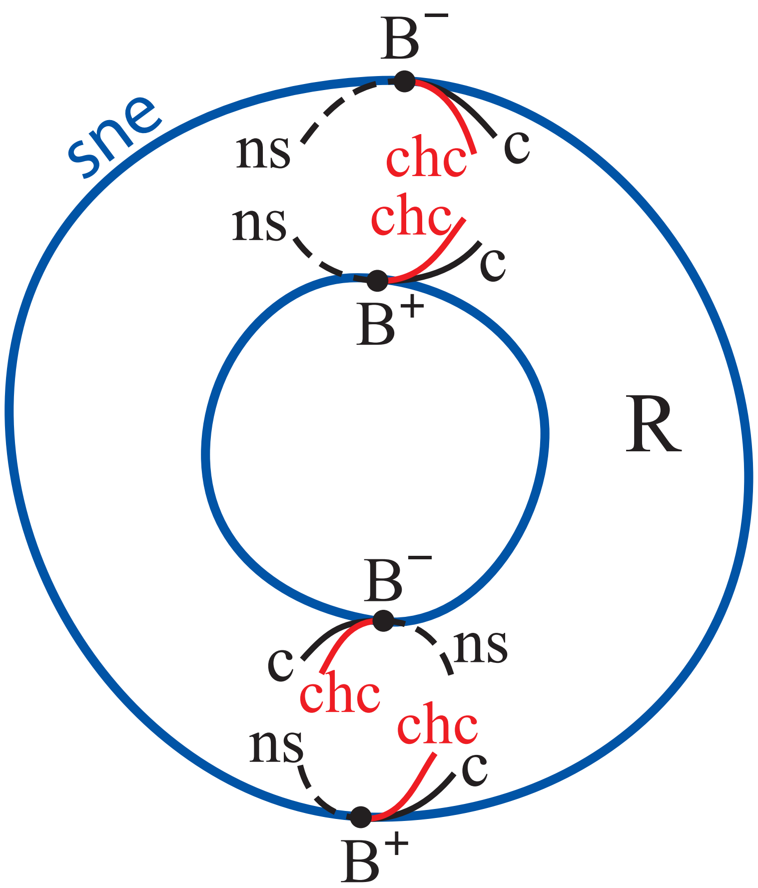
Label B points by the index of the equilibrium to which $\delta x'$ points.

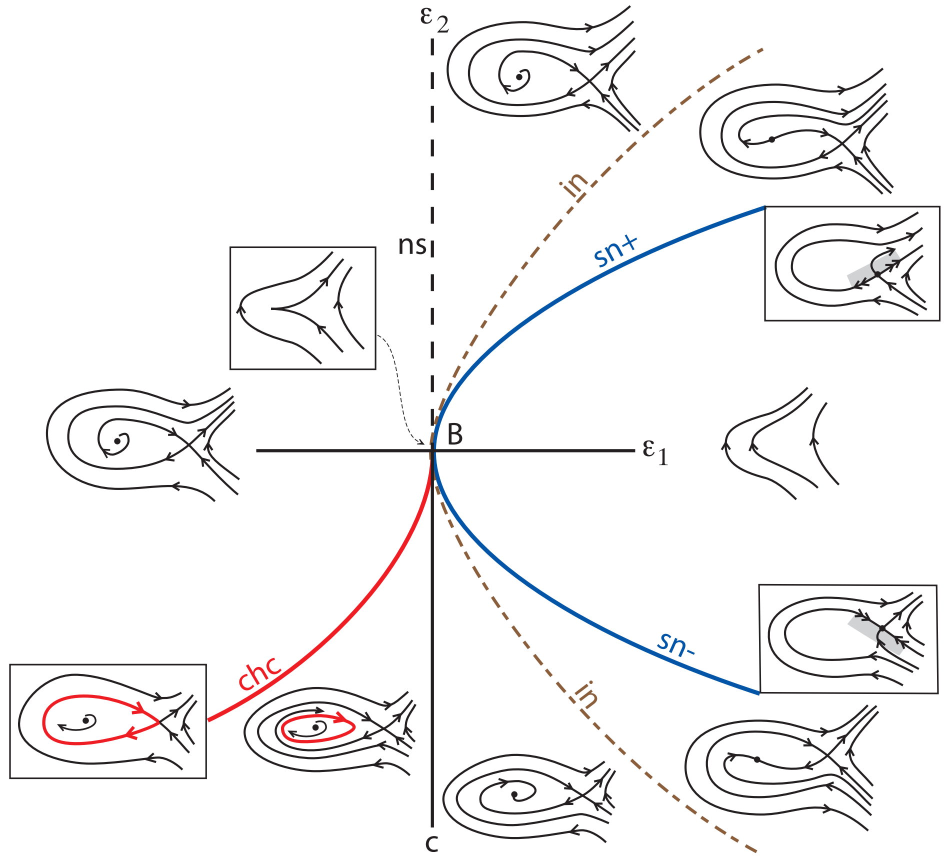
(= Fiedler's B-index).

So each sne curve contains at least one B^+ and one B^- point.

- Assume **precisely four** B **points.**

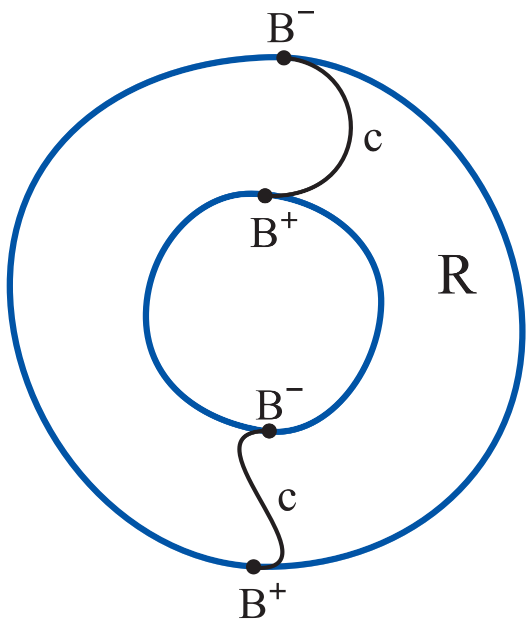
B points produce **arcs of centre** (c), **neutral saddle** (ns), and **contractible homoclinic connection** (chc), and a **contractible periodic orbit** (cpo) between the c and chc curves.





3. Trace-zero curves: We extended proof of B points – there exist at least two curves of centre connecting B points on opposite saddle-node curves.

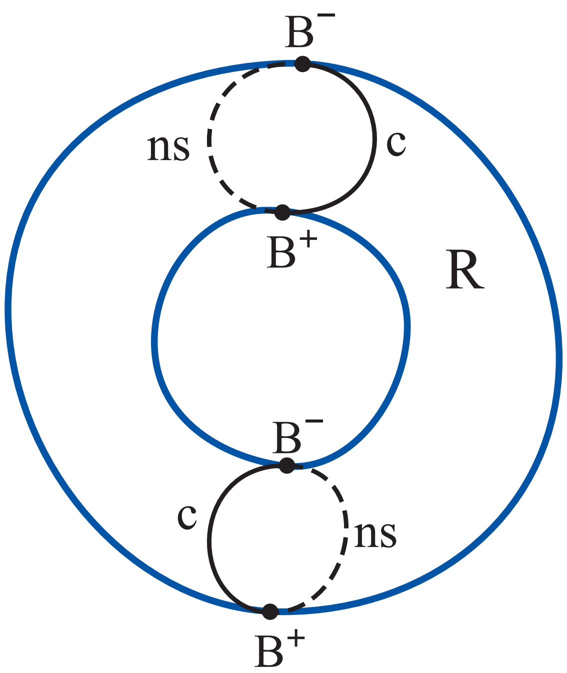
With 4 B points we have **precisely two such**. Fiedler showed that **a curve of centre from a B point must join to a B point of opposite B-index**. They cannot cross, so up to orientation of tangencies:



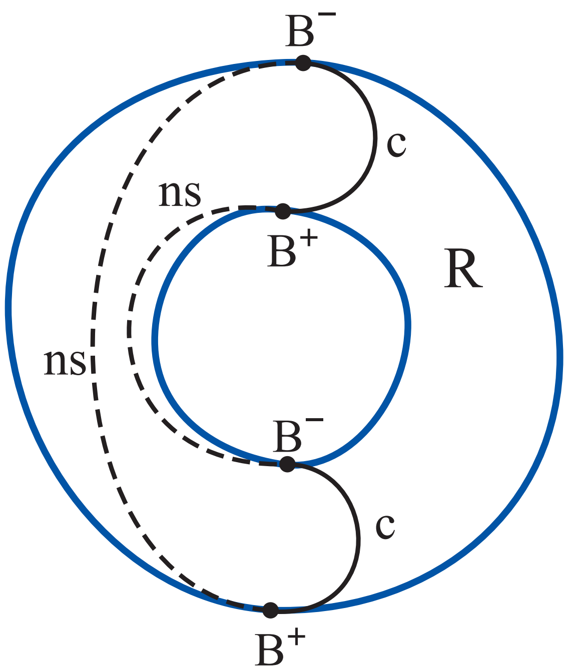
Trace-zero curves either connect B points or form closed curves avoiding sne. Assume none of the latter.

Various options for neutral-saddle curves, but intersections of ns with c make extra bifurcations so assume no intersections.

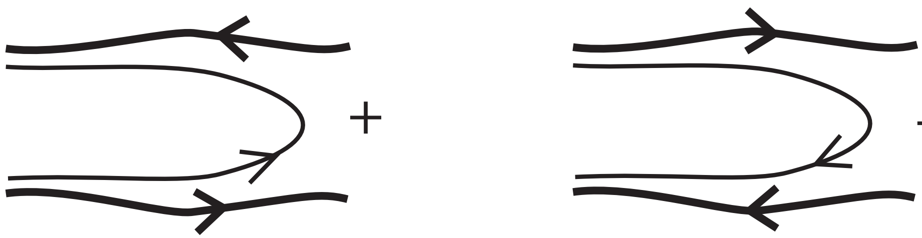
End up with only (up to orientations):



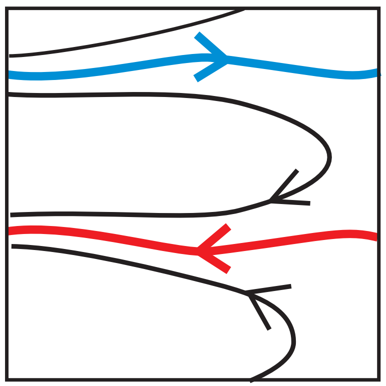
and



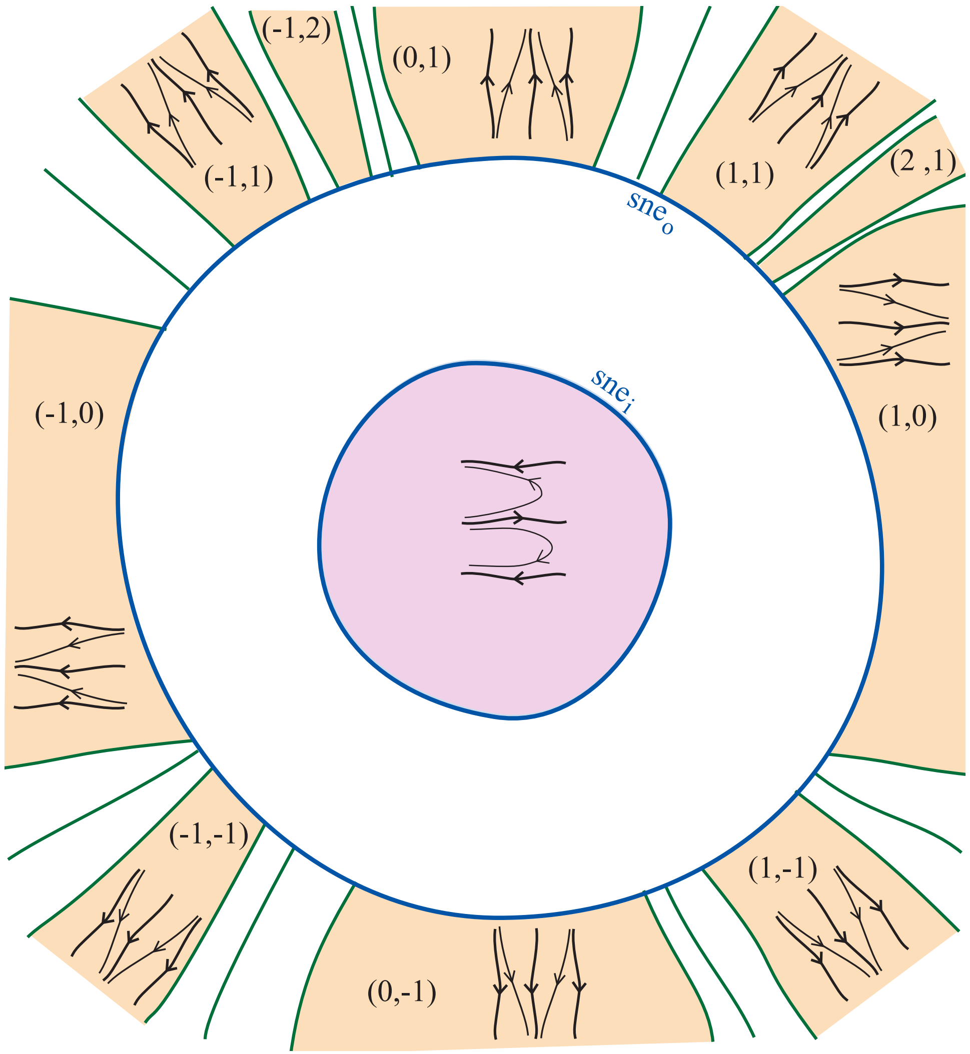
4. Flow in hole: As x performs one revolution of type (m, n), $G(\Omega, x)$ performs n revolutions. Deduce there exists no cross-section to the flow, hence at least two Reeb components of horizontal type, with $G(\Omega, x)$ making a total of one rotation for one vertical revolution of x.



- Assume minimum number of invariant annuli, then must be



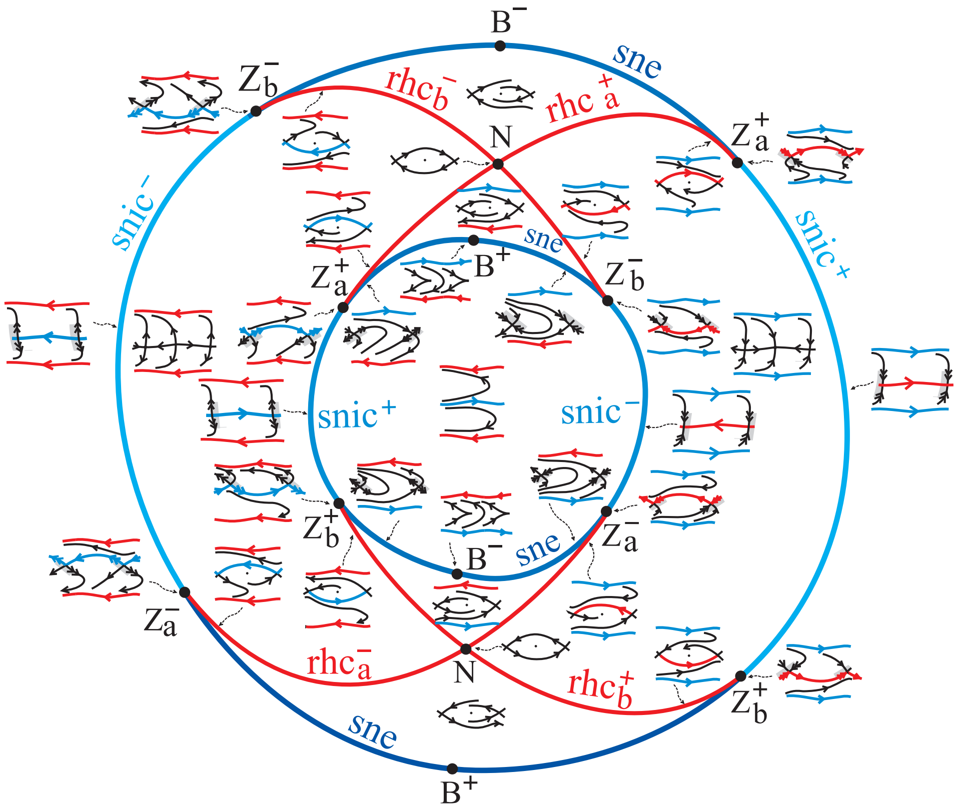
5. Flow outside R: When x makes any revolution, $G(\Omega, x)$ makes no rotations. Assume flow has no Reeb components. Then there is a cross-section (Poincaré flow), so every orbit has the same direction of average velocity (homology direction).

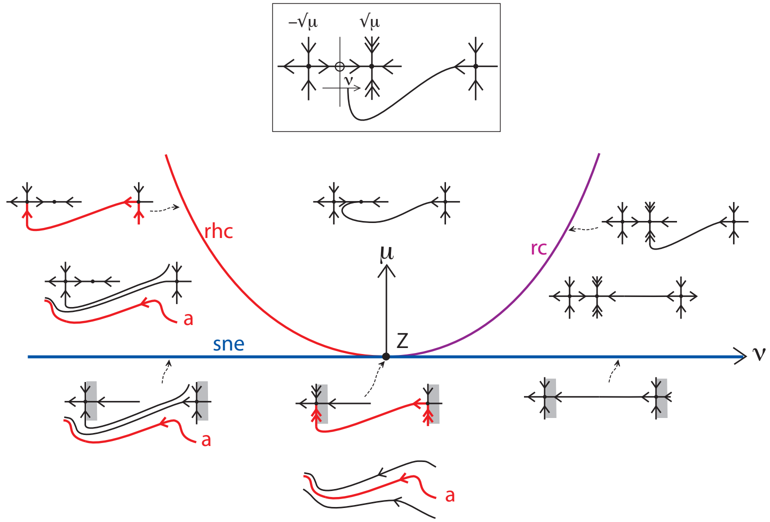


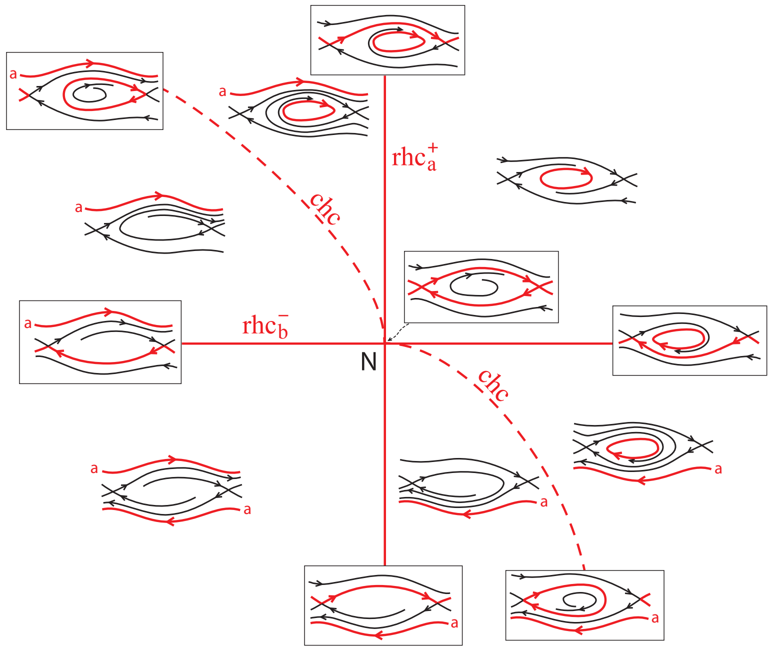
The homology direction makes one revolution as Ω makes one revolution outside R.

6. Saddle-node loops (Z points), rotational homoclinic connections (rhc), and necklace (N points):

A rather long argument leads to at least: (assume no more)

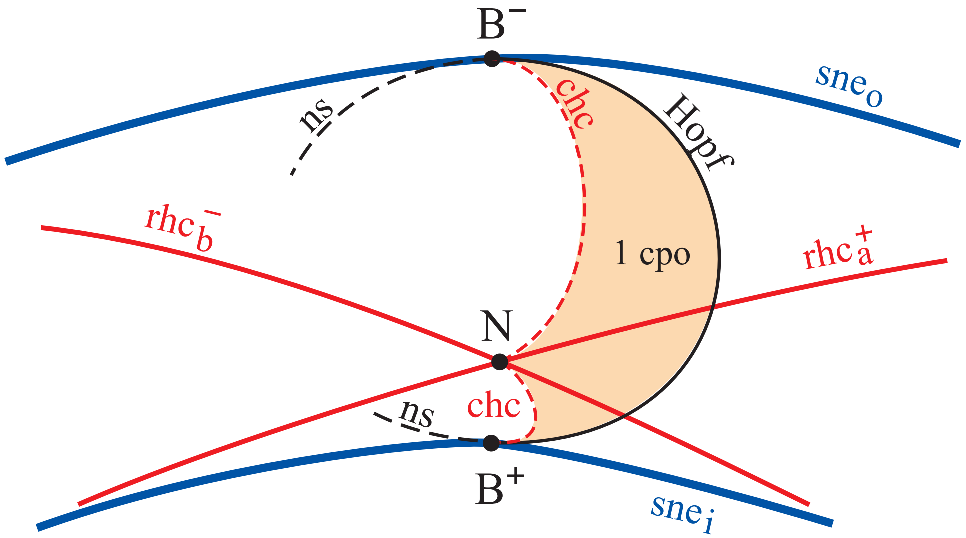


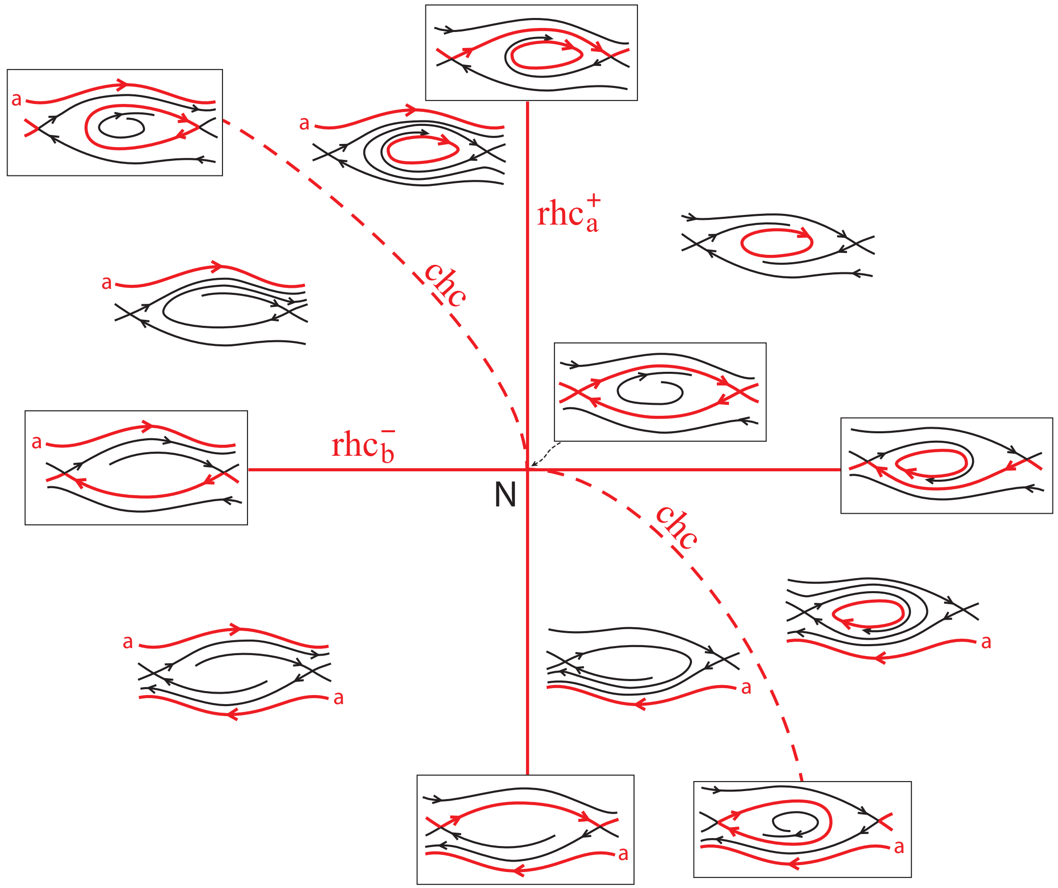




7. Contractible periodic orbits (cpo):

Both B and N points generate curves of contractible homoclinic connection (chc). Assume at most one contractible periodic orbit then:

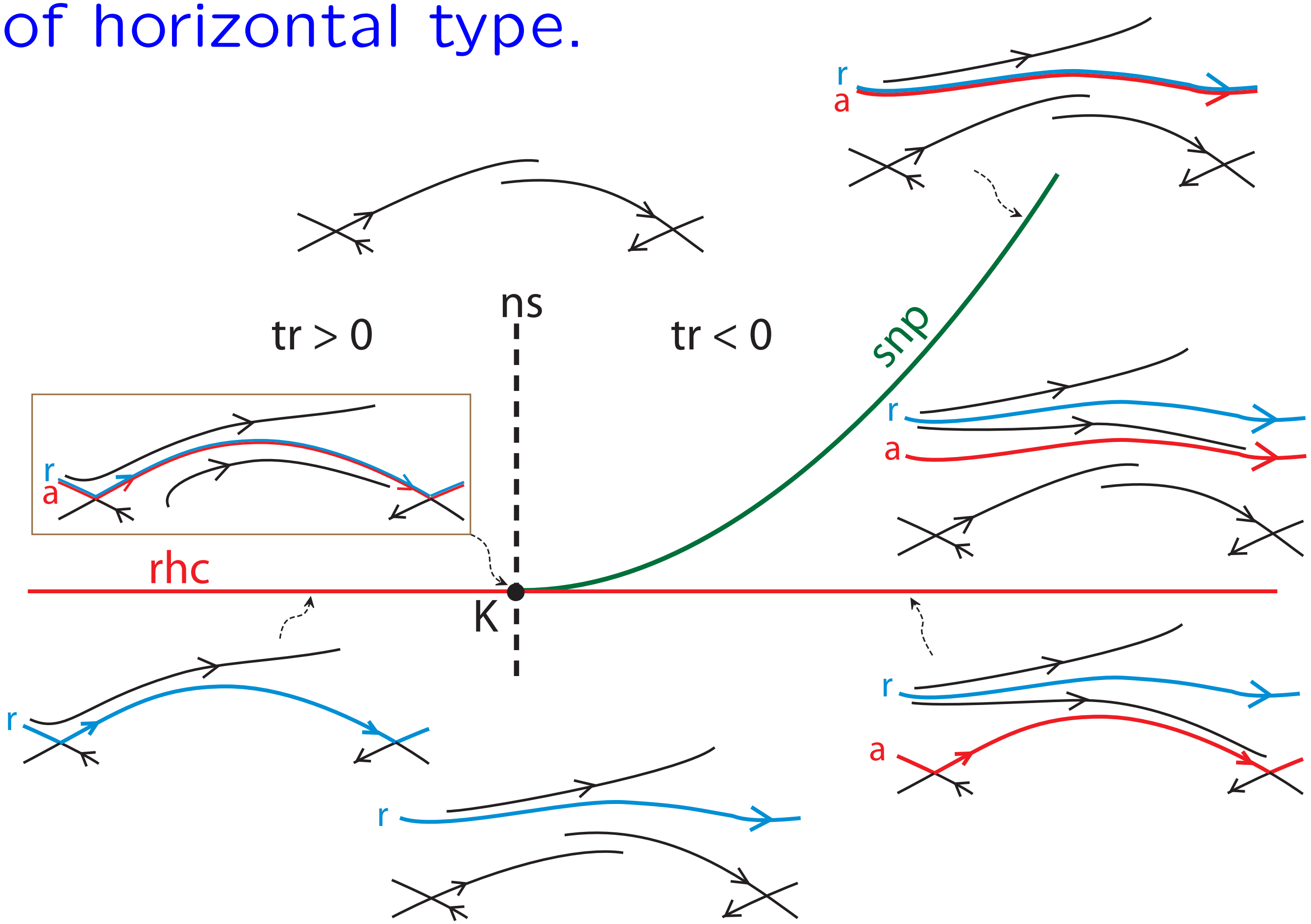




8. Neutral rotational homoclinic connections (K point):

The ns curves must intersect the rhc curves. This implies at least two K points of horizontal type in each of the top and bottom of R.

Assume no more. They generate curves of saddle-node periodic orbit of horizontal type.



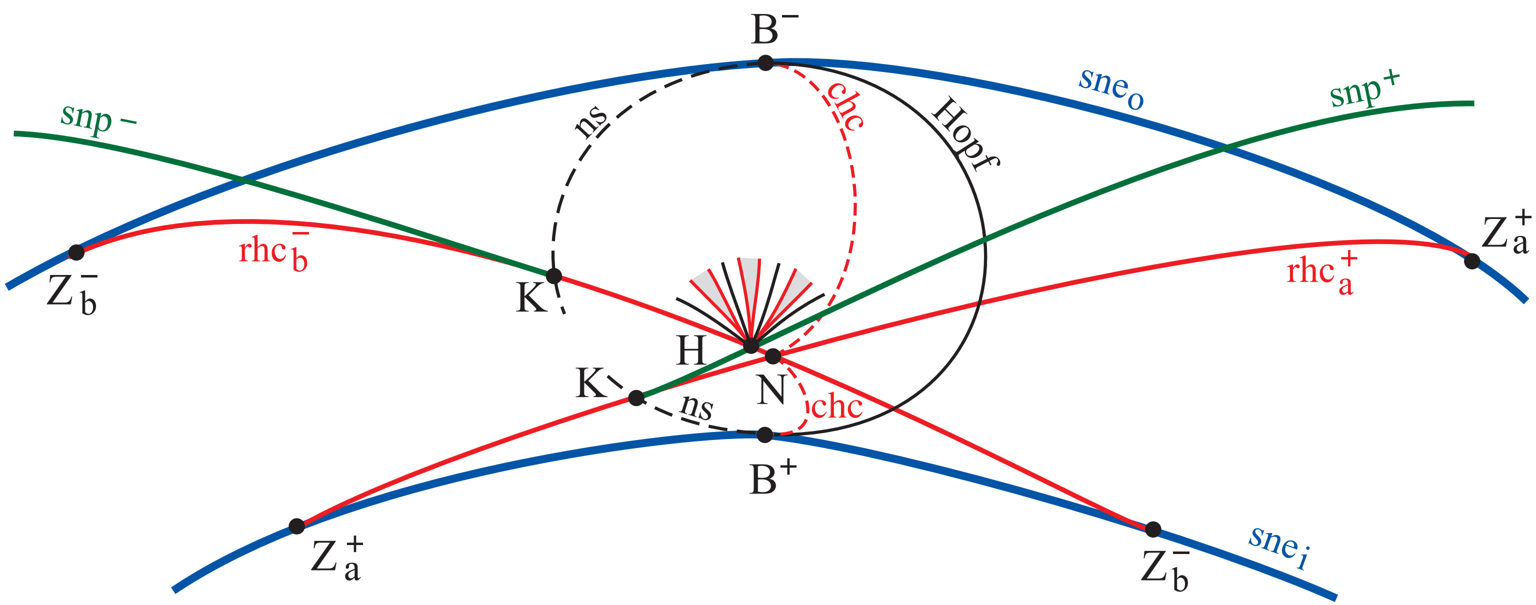
9. "Half-plane fan" (H point):

The snp curves from the K points have to join to snp curves outside R, hence at least one creates an H point ($\mathrm{snp}^+ \cap \mathrm{rhc}^-$).

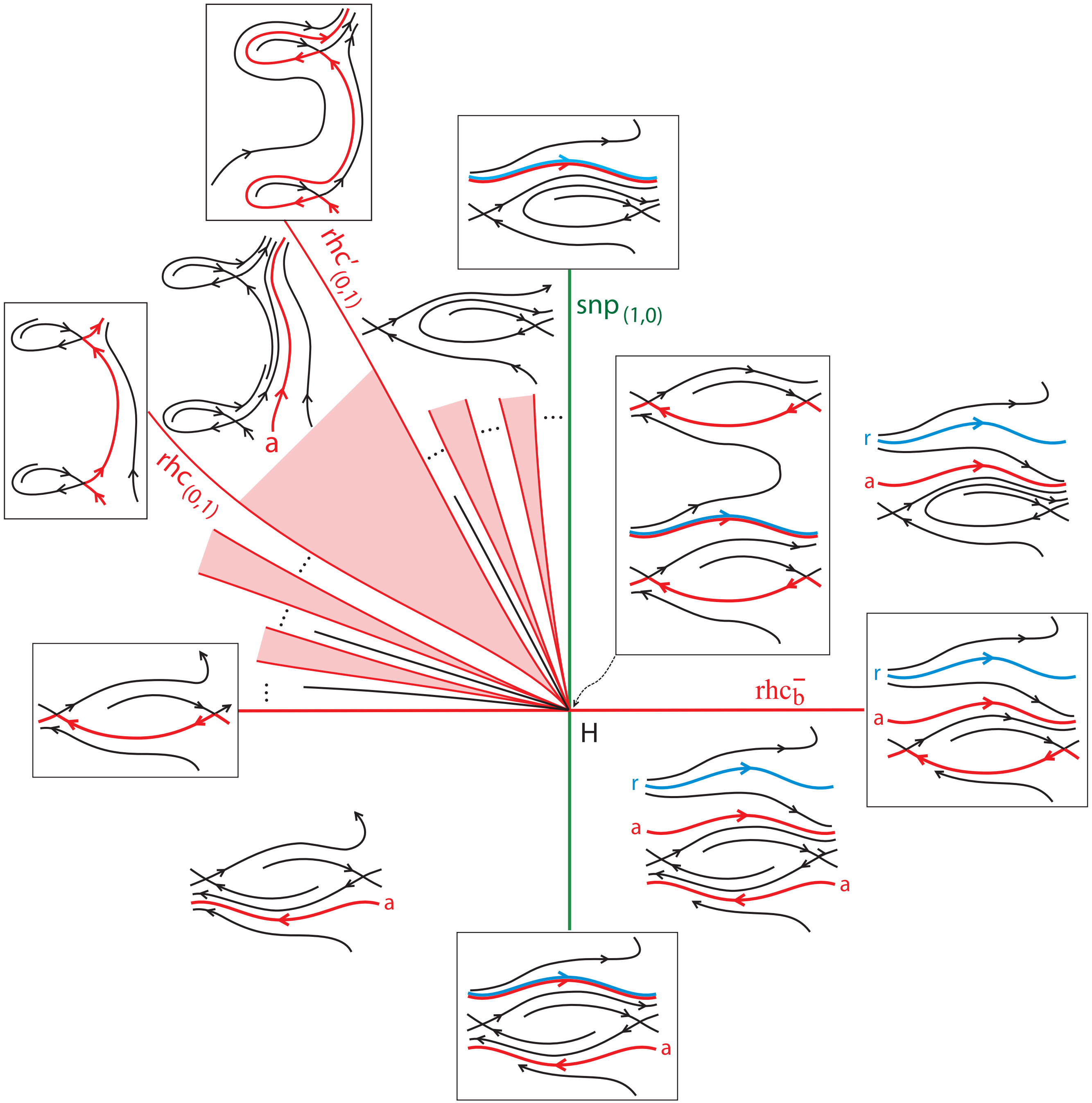
Assume just one at the top and one at the bottom.

An H point generates a half-plane fan of tongues for each rational homology direction and curves for each irrational homology direction in a half plane.

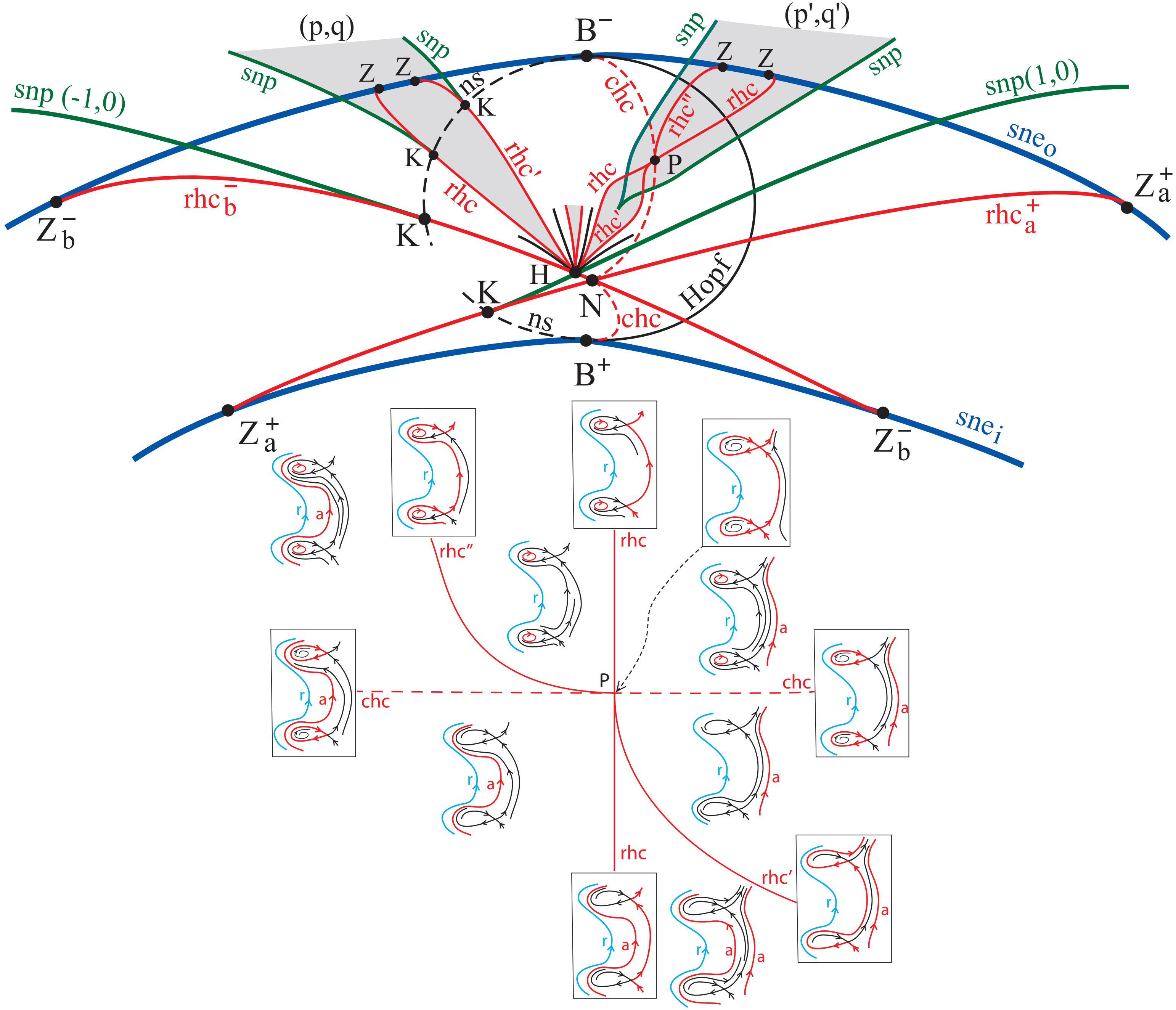
The rational tongues all start with rhc boundaries.

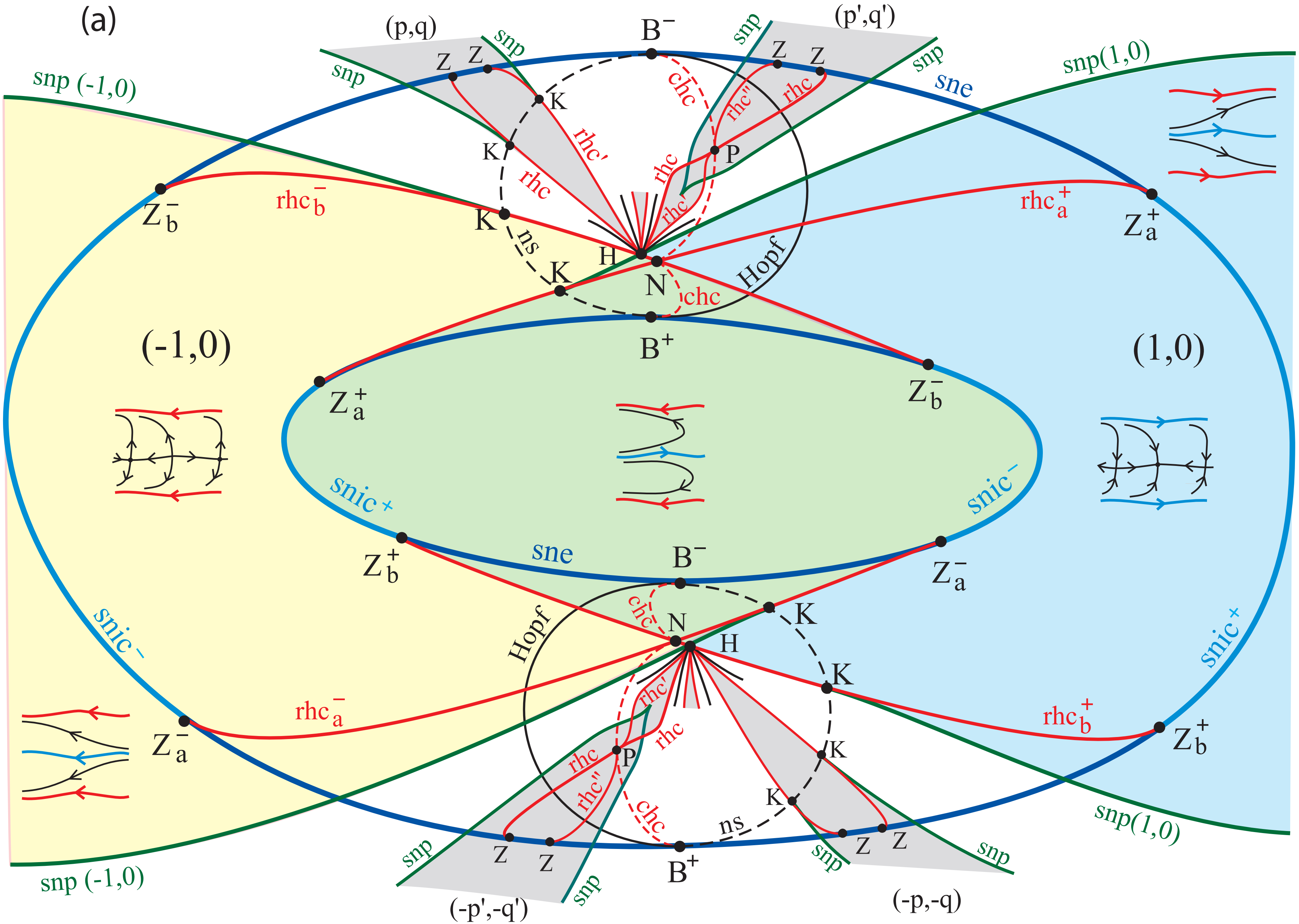


Unfolding of a Half-plane fan [BGKM91]:

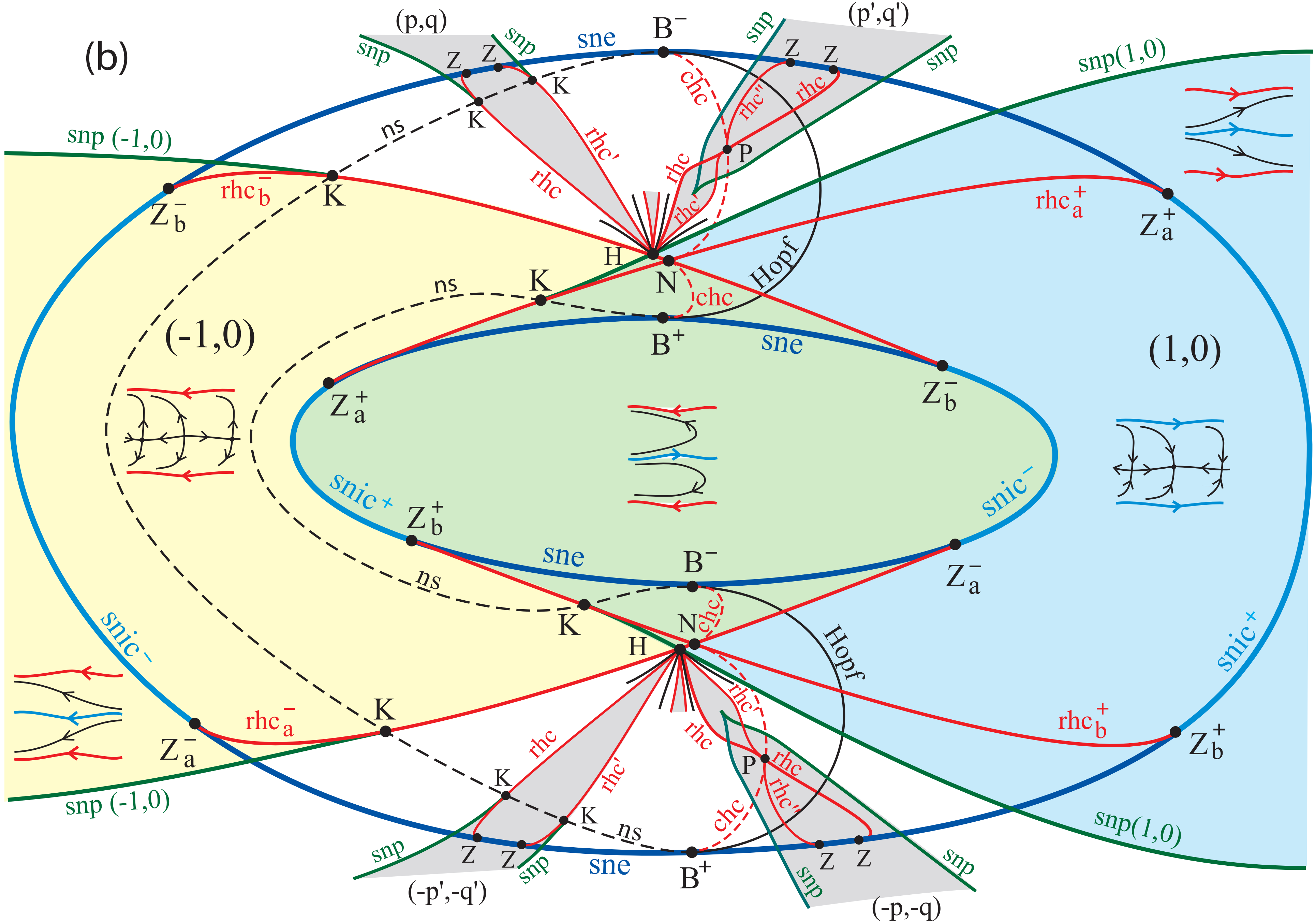


To complete the diagram, analyse intersection of tongues with ns and chc, which create K points and pendant points (P).





(a)
(p,q)
(p',q')
B$^-$
B$^+$
snp
snp (-1,0)
snp(1,0)
sne
Z
K
P
rhc
rhc'
rhc''
chc
rhc$_b^-$
rhc$_a^+$
rhc$_a^-$
rhc$_b^+$
Z$_b^-$
Z$_a^+$
Z$_b^+$
Z$_a^-$
ns
H
N
Hopf
(-1,0)
(1,0)
snic$^+$
snic$^-$
(-p',-q')
(-p,-q)



(b)
(p,q)
(p',q')
B⁻
sne
snp
Z
K
chc
rhc''
rhc
P
snp(1,0)
snp (-1,0)
ns
rhc'
rhc
Z_b^-
rhc_b^-
K
rhc_a^+
Z_a^+
H
N
Hopf
(-1,0)
B⁺
sne
(1,0)
Z_a^+
Z_b^-
snic⁺
snic⁻
Z_b^+
sne
B⁻
Z_a^-
ns
K
N
chc
H
Hopf
snic⁻
snic⁺
K
rhc_a^-
rhc_b^+
Z_a^-
Z_b^+
snp (-1,0)
snp(1,0)
(-p',-q')
(-p,-q)

We prove for ε small our example satisfies all our minimality assumptions except it has a region with more than one cpo.

Reference

C Baesens, RS MacKay, Simplest bifurcation diagrams for monotone families of vector fields on a torus, Nonlinearity 31 (2018) 2928–81

Project for the future: Genus > 1. Then all three cycles can have non-zero homology.

Joyeux Anniversaire, Jean-Marc !