Maximal isotopies, transverse foliations and orbit forcing theory for surface homeomorphisms

Patrice Le Calvez (IMJ-PRG)
join work with Fabio Tal (USP)

Nice
Systèmes dynamiques et systèmes complexes
(pour célebrer les 60 ans de Jean-Marc)

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Forcing theory in dim 1

Theorem [Sharkovski]: *Define the following order* $\lhd$ *on* $\mathbb{N} \setminus \{0\}$*:*

$$1 \lhd 2 \lhd 2^2 \lhd \dots 2^2.5 \lhd 2^2.3 \lhd \cdots \lhd 2.5 \lhd 2.3 \lhd \cdots \lhd 7 \lhd 5 \lhd 3$$

A continuous map $f : \mathbb{R} \to \mathbb{R}$ *with a periodic point of period* p *has a periodic point of period* q*, if* $q \lhd p$*.*

Theorem [Li-Yorke]: *If a continuous map* $f : \mathbb{R} \to \mathbb{R}$ *has a periodic point of period* 3*, then*

- *The topological entropy of* $h(f)$ *is positive;*
- *There exists* $\lambda > 1$ *such that* $\sharp\mathrm{Fix}(f^n) \geqslant \lambda^n$*, for every* $n \geqslant 1$*.*

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A continuous map $f : \mathbb{R} \to \mathbb{R}$ with a periodic point of period p has a periodic point of period q, if $q \lhd p$.

Theorem [Li-Yorke]: *If a continuous map $f : \mathbb{R} \to \mathbb{R}$ has a periodic point of period 3, then*

- *The topological entropy of $h(f)$ is positive;*
- *There exists $\lambda > 1$ such that $\sharp\mathrm{Fix}(f^n) \geqslant \lambda^n$, for every $n \geqslant 1$.*

- true if there exists x such that $f^3(x) \leqslant x < f(x) < f^2(x)$
- f has a topological horsehoe
- key tool: intermediate value theorem

Topological horseshoe

Definition: Let f be a continuous map on a topological space X. Say that f has a *topological horseshoe* if there exists a compact set $Y \subset X$ invariant by an iterate f^r, $r \geqslant 1$, such that:

- $(Y, f^r|_Y)$ is an extension of the Bernouill shift $(\sigma, \{0,1\}^{\mathbb{N}})$;
- every periodic sequence $e \in \{0,1\}^{\mathbb{Z}}$ admits at least one periodic point of f^r with same period in its fiber;

(replace $\{0,1\}^{\mathbb{N}}$ with $\{0,1\}^{\mathbb{Z}}$ if f is a homeomorphism).

Remarks:

- The entropy of f is positive and the set of periodic points, for an iterate, grows exponentially fast with the period;
- If M is a surface and f is a C^2-diffeomorphism, it is equivalent to have positve entropy (Katok).

What happens in dimension 2 ?

Forcing theories exist for surface homeomorphisms:

- for braid types of periodic orbits (Nielsen -Thurston theory);
- Homotopic Brouwer theory (Franks-Handel).

We present a forcing theory that works for segments of orbits. It is not explicit but has many applications, some related to rotation sets, some related the existence of topological horseshoes

- objects: paths in the space of leaves of a certain foliation
- key tool: intersection lemma

Transitive homeomorphisms of the sphere $\mathbb{S}^2$

We have the following generalization of a result of M.Handel.

Theorem [LT]: *If f is an orientation preserving homeomorphism of $\mathbb{S}^2$ with a dense orbit. Then:*

- *either f has a topological horseshoe;*
- *or f is an irrational pseudo rotation.*

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Definition : An irrational pseudo rotation is an orientation preserving homeomorphism f of $\mathbb{S}^2$ such that

- f has no periodic point but two fixed points N, S;
- every other point turns around N and S with irrational angular speed.

(Anosov-Katok examples)

Transitive sets of entropy zero homeomorphism of $\mathbb{S}^2$

Theorem [LT]: *Let f be a homeomorphism of $\mathbb{S}^2$ with no topological horseshoe. If X is a closed invariant set with a dense orbit, then:*

1. *either X is a periodic orbit;*
2. *or X has irrational type;*
3. *or $f_{|X}$ is an extension of an odometer.*

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In case 2. one can find irrational invariant curves, Aubry-Mather sets, hedgehogs, irrational pseudo rotation,...

In case 3. there exists a nested sequence $(D_n)_{n\geqslant 0}$ of open disks and an increasing sequence $(q_n)_{n\geqslant 0}$ of integers with $q_n|q_{n+1}$ such that

$$f^{q_n}(D_n) = D_n, \ \ f^k(D_n) \cap D_n = \emptyset \text{ if } 0 < k < q_n,$$

$$X \subset \bigcup_{0\leqslant k<q_n} f^k(D_n).$$

Homological intersection

Theorem **[Lellouch]:** *Let M be a closed surface of genus $g \geqslant 2$ and f a homeomorphism of M isotopic to the identity. Suppose that there exist two Borel invariant probability measures whose rotation vectors have a non vanishing intersection number, then f has a topological horseshoe.*

Definition: The rotation vector $\rho(\mu) \in H_1(M, \mathbb{R})$ of an invariant measure μ is defined as follows:

$$\int_M \int_{I(z)} \omega \, d\mu(z) = \langle [\omega], \rho(\mu) \rangle, \text{ where}$$

- $I = (f_t)_t \in [0,1]$ is an isotopy from Id to f and $I(z) : t \mapsto f_t(z)$ the trajectory of z;
- ω is a closed 1-form and $[\omega] \in H^1(M, \mathbb{R})$ its cohomology class.

Generalization of results of Katok, Llibre-MacKay, Pollicott, Matsumoto.

Maximal isotopies (I)

Let M be an oriented surface and f a homeomorphism of M isotopic to the identity. We define

$$\mathcal{E} = \{\text{isotopies } \; I = (f_t)_{t\in[0,1} \; \text{from Id to } f\}.$$

Definitions: For every $I = (f_t)_{t\in[0,1} \in \mathcal{E}$, one can define:

- the *trajectory* $I(z) : t \mapsto f_t(z)$ of $z \in M$;
- the *whole trajectory* $I^{\mathbb{Z}}(z) = \prod_{k\in\mathbb{Z}} I(f^k(z))$ of $z \in M$;
- the *fixed point set* $\mathrm{fix}(I) = \bigcap_{t\in[0,1]} \mathrm{fix}(f_t)$;
- the *domain* $\mathrm{dom}(I) = M \setminus \mathrm{fix}(I)$.

Maximal isotopies (II)

Here, M is an oriented surface, f a homeomorphism of M and $\mathcal{E}$ the (non empty) set of isotopies from Id to f.

Definition: Say that $I \in \mathcal{E}$ is *smaller* than $I' \in \mathcal{E}$ ($I \preceq I'$) if

- $\mathrm{fix}(I) \subset \mathrm{fix}(I')$;
- I' is homotopic to I relative to $\mathrm{fix}(I)$.

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Theorem [Béguin-Crovisier-Le Roux, Jaulent]: *For every $I \in \mathcal{E}$, there exists $I' \in \mathcal{E}$ such that $I \preceq I'$ and such that I' is maximal.*

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- $I \in \mathcal{E}$ is maximal iff for every $z \in \mathrm{fix}(f) \setminus \mathrm{fix}(I)$, the loop $I(z)$ is non homotopic to zero in $\mathrm{dom}(I)$.

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Remarks:

- $I \in \mathcal{E}$ is maximal iff for every $z \in \mathrm{fix}(f) \setminus \mathrm{fix}(I)$, the loop $I(z)$ is non homotopic to zero in $\mathrm{dom}(I)$.
- if $(f_t)_{t\in\mathbb{R}}$ is a flow, then $I = (f_t)_{t\in[0,1]}$ is maximal.

Transverse foliations (I)

Here, M is an oriented surface, f a homeomorphism of M and $\mathcal{E}$ the (non empty) set of isotopies from Id to f.

Theorem [L]: *If $I \in \mathcal{E}$ is maximal, there exists a singular oriented C^0- foliation $\mathcal{F}$ on M such that:*

- $\mathrm{sing}(\mathcal{F}) = \mathrm{fix}(I)$;
- *every trajectory $I(z)$, $z \in \mathrm{dom}(I)$, is homotopic to a path γ transverse to $\mathcal{F}$, meaning that γ crosses (locally) every leaf from the right to the left.*

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Remark: If I is induced by a flow defined by a vector field ξ, one can choose $\mathcal{F}$ as the set of integral curves of η, where $\eta \wedge \xi > 0$.

Transverse foliations (II)

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Remark: If $\widetilde{I} = (\widetilde{f}_t)_{t\in[0,1]}$ is the isotopy on the universal covering space $\widetilde{\mathrm{dom}}(I)$ of $\mathrm{dom}(I)$ starting from Id that lifts $(f_{t|\mathrm{dom}(I)})_{t\in[0,1]}$, then every leaf ϕ of the lifted foliation $\widetilde{\mathcal{F}}$ is a *Brouwer line* of $\widetilde{f}_1$: it separates $\widetilde{f}_1(\phi)$ which is on its left and $\widetilde{f}_1^{-1}(\phi)$ which is on its right.

Transverse foliations (III)

Theorem [L]: *If $I \in \mathcal{E}$ is maximal, there exists a singular oriented C^0- foliation $\mathcal{F}$ on M such that:*

- $\mathrm{sing}(\mathcal{F}) = \mathrm{fix}(I)$;
- *every trajectory $I(z)$, $z \in \mathrm{dom}(I)$, is homotopic to a path γ transverse to $\mathcal{F}$, meaning that γ crosses (locally) every leaf from the right to the left.*

Definitions: The path γ is defined up to a natural relation of equivalence. For every $z \in \mathrm{dom}(I)$ one can define:

- the *transverse trajectory* $I_{\mathcal{F}}(z) = \gamma$;
- the *whole transverse trajectory* $I_{\mathcal{F}}^{\mathbb{Z}}(z) = \prod_{k \in \mathbb{Z}} I_{\mathcal{F}}(f^k(z))$.

Fundamental result

Proposition [LT]: *Let f be a homeomorphism of an oriented surface M isotopic to the identity, I a maximal isotopy of f and $\mathcal{F}$ a transverse foliation. If f has no topological horseshoe, the whole transverse trajectory $I_{\mathcal{F}}^{\mathbb{Z}}(z)$ of every point $z \in \mathrm{dom}(I)$ has no* $\mathcal{F}$-transverse self intersection*: it may be chosen as a simple curve or a loop.*

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Remark: If f has no topological horseshoe, the transverse trajectories look like orbits of a flow and the dynamics of f itself look like the dynamics of a flow. For instance, if $M = \mathbb{S}^2$ and z is recurrent, then $I_{\mathcal{F}}^{\mathbb{Z}}(z)$ may be chosen as a loop (there exists an adequate Poincaré-Bendixson theory).

$\mathcal{F}$-transverse intersection

Let $\widetilde{\mathcal{F}}$ be a C^0 non singular oriented foliation on $\mathbb{R}^2$.

- Every leaf ϕ is an embedded oriented proper line separating $\mathbb{R}^2$ into $R(\phi)$ (on the right) and $L(\phi)$ (on the left).
- The space of leaves is an oriented simply connected manifold (usually non Hausdorff) of dimension 1 (Haefliger-Reeb).

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Definition: Say that two transverse paths $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ defined on $\mathbb{R}$ have a *$\widetilde{\mathcal{F}}$-transverse intersection* if $\exists a_1 < b_1$, $\exists a_2 < b_2$ such that:

- $\phi_{\widetilde{\gamma}_1(a_1)} \subset L(\phi_{\widetilde{\gamma}_2(a_2)})$, $\phi_{\widetilde{\gamma}_2(a_2)} \subset L(\phi_{\widetilde{\gamma}_1(a_1)})$;
- $\phi_{\widetilde{\gamma}_1(b_1)} \subset R(\phi_{\widetilde{\gamma}_2(b_2)})$, $\phi_{\widetilde{\gamma}_2(b_2)} \subset R(\phi_{\widetilde{\gamma}_1(b_1)})$;
- $\exists$ a "crossing" between $\phi_{\widetilde{\gamma}_1(a_1)}$, $\phi_{\widetilde{\gamma}_2(a_2)}$, $\phi_{\widetilde{\gamma}_1(b_1)}$, $\phi_{\widetilde{\gamma}_2(b_2)}$.

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Definition: A transverse path γ to a non singular oriented foliation $\mathcal{F}$ on a surface M has a $\mathcal{F}$-transverse self intersection if there exists two lifts of γ to the universal covering space $\widetilde{M}$, with a $\widetilde{\mathcal{F}}$-transverse intersection.