Quasipatterns in the superposition of two hexagonal patterns for the Swift-Hohenberg PDE

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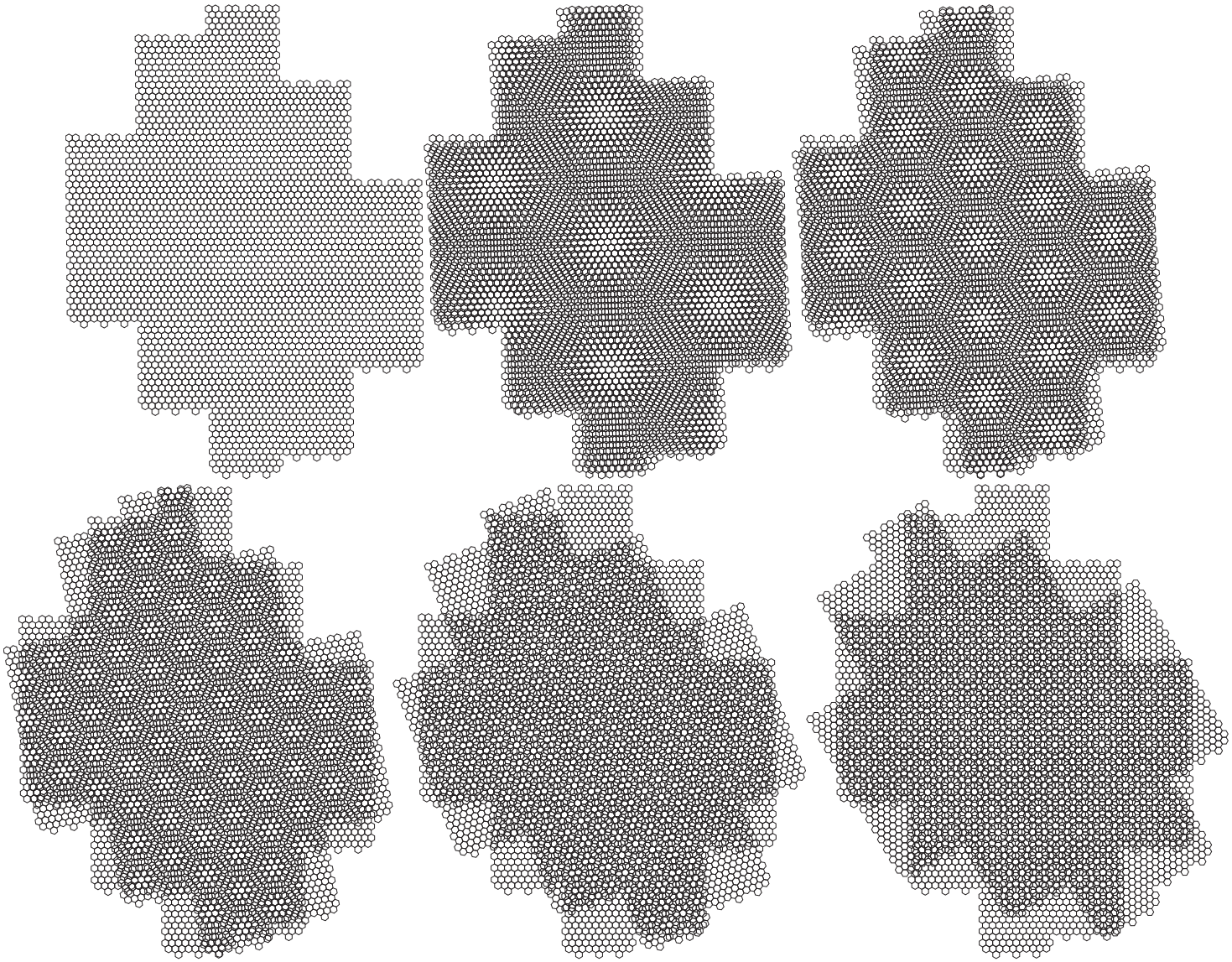
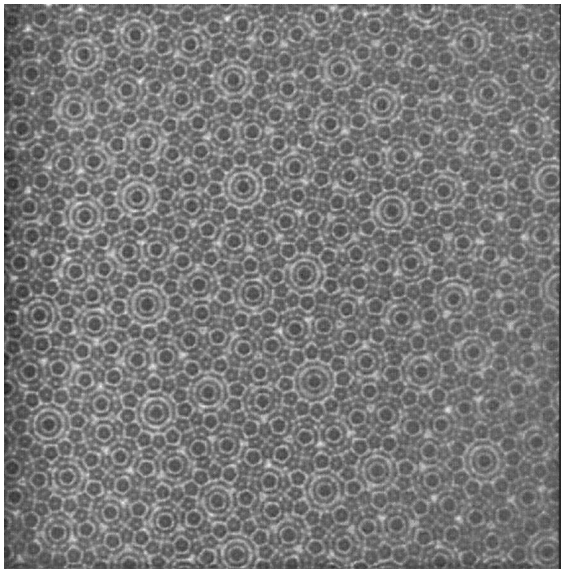
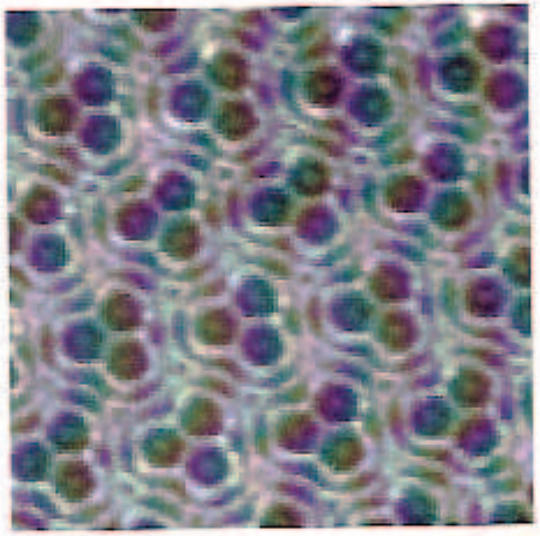
Superposition experiments



Superposition of two hexagonal patterns: $0^o, 3^o, 5^o, 10^o, 20^o, 30^o$

Quasipatterns experiments





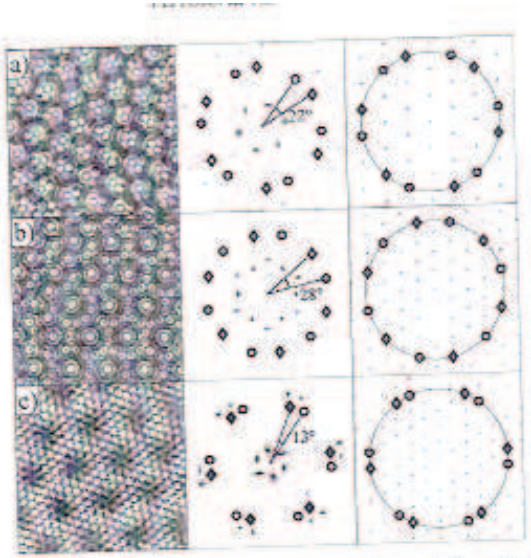


FIG. 3. (a) 3:2, (b) 4:3, and (c) 5:3 grid states. The spatial spectra (center) corresponding to the photographs (left) are spanned by grids (right) whose basis vectors, $\vec{k}_{grid}$, are stable modes. Fluid parameters are as in Fig. 2. Fluid depths: (a) and (c) 0.3 cm and (b) 0.2 cm. ○ and denote corotated sets of $\vec{K}_c$.

Experiments of Faraday type. Kudrolli, Pier, Gollub 1998, Epstein, Fineberg 2006

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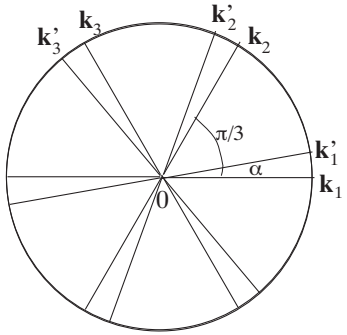
Definition of the quasilattice

$$\Gamma = \{\mathbf{k} \in \mathbb{R}^2; \mathbf{k} = \sum_{j=1,\ldots 6} m_j \mathbf{k}_j + m'_j \mathbf{k}'_j,\ \ m_j, m'_j \in \mathbb{N}\}.$$



Special angles $\mathcal{E}_p := \{\alpha \in \mathbb{R}/2\pi\mathbb{Z}; \cos\alpha \in \mathbb{Q}, \cos(\alpha + \pi/3) \in \mathbb{Q}\}$.

Lemma

The set $\mathcal{E}_p$ has a zero measure in $\mathbb{R}/2\pi\mathbb{Z}$.
(i) If the wave vectors $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}'_1, \mathbf{k}'_2$ are not independent on $\mathbb{Q}$, then $\alpha \in \mathcal{E}_p$.
(ii) If $\alpha \in \mathcal{E}_p$ then the lattice Γ is periodic with an hexagonal symmetry, and wave vectors $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}'_1, \mathbf{k}'_2$ are combinations of only two smaller vectors, of equal length making an angle $2\pi/3$.

Fourier series

$u(x,y)$ function under the form of a Fourier expansion

$$u = \sum_{\mathbf{k}\in\Gamma} u^{(\mathbf{k})} e^{i\mathbf{k}\cdot\mathbf{x}}, \;\; u^{(\mathbf{k})} = \overline{u}^{(-\mathbf{k})} \in \mathbb{C}. \tag{1}$$

$\mathbf{k} \in \Gamma$ may be written as

$$\mathbf{k} = x_1\mathbf{k}_1 + x_2\mathbf{k}_2 + x_3\mathbf{k}_1' + x_4\mathbf{k}_2', \quad (x_1, x_2, x_3, x_4) \in \mathbb{Z}^4,$$

If $\alpha \in \mathcal{E}_{qp} = \mathcal{E}_p^c$, , Γ spans a 4-dimensional vector space on $\mathbb{Q}$.

$$N_{\mathbf{k}} = \sum_{j=1,\ldots,4} |x_j| \text{ is a norm of } \mathbf{k}$$

Hilbert spaces $\mathcal{H}_s$, $s \geq 0$:

$$\mathcal{H}_s = \left\{ u = \sum_{\mathbf{k}\in\Gamma} u^{(\mathbf{k})} e^{i\mathbf{k}\cdot\mathbf{x}}; u^{(\mathbf{k})} = \overline{u}^{(-\mathbf{k})} \in \mathbb{C}, \;\; \sum_{\mathbf{k}\in\Gamma} |u^{(\mathbf{k})}|^2 (1 + N_{\mathbf{k}}^2)^s < \infty \right\},$$

basic Lemmas

Lemma

If $\alpha \in \mathcal{E}_{qp}$, a function defined by a convergent Fourier series as above represents a quasipattern, i.e. is quasiperiodic in all directions.

Lemma

For nearly all $\alpha \in (0, \pi/6)$, in particular for $\alpha \in \mathbb{Q}\pi \cap (0, \pi/6]$, the only solutions of $|\mathbf{k}(\mathbf{x})| = 1$ are $\pm\mathbf{k}_j, \pm\mathbf{k}'_j$ $j = 1, 2$ and $\mathbf{k} = \pm\mathbf{k}_3$, or $\pm\mathbf{k}'_3$, i.e. corresponding to $\mathbf{x} = (\pm 1, \mp 1, 0, 0)$ or $(0, 0, \pm 1, \mp 1)$.

$\mathcal{E}_0$ is the set of α's such that Lemma above applies.

Lemma

For nearly all $\alpha \in \mathcal{E}_{qp} \cap (0, \pi/6)$, and for $\varepsilon > 0$, there exists $c > 0$ such that, for all $|\mathbf{x}| > 0$ such that $|\mathbf{k}(\mathbf{x})| \neq 1$, $(|\mathbf{k}(\mathbf{x})|^2 - 1)^2 \geq \frac{c}{|\mathbf{x}|^{12+\varepsilon}}$ holds.

Steady Swift-Hohenberg equation in $\mathbb{R}^2$

$$(1+\Delta)^2 u = \mu u - u^3,\ \mathbf{x} \in \mathbb{R}^2 \to u(\mathbf{x}) \in \mathbb{R}$$

$$e^{i\mathbf{k}.\mathbf{x}} \in Ker\{(1+\Delta)^2 - \mu\} \text{ in } \mathcal{H}_s$$

iff Dispersion equation holds: $(1-|\mathbf{k}|^2)^2 = \mu,\ \mathbf{k} \in \Gamma$

For $\mu = 0$ all wave vectors $\mathbf{k}$ with $|\mathbf{k}| = 1$ are **critical**

We choose to look for solutions in $\mathcal{H}_s$, for $\alpha \in \mathcal{E}_{qp} \cap \mathcal{E}_0$, i.e. quasiperiodic in $\mathbb{R}^2$, moreover invariant under rotations of angle $\pi/3$ and bifurcating for μ close to 0.

define $\mathbf{L}_0 = (1+\Delta)^2$

For $\alpha \in \mathcal{E}_0$ $Ker\mathbf{L}_0$ is 2-dimensional spanned by

$$v = \sum_{j=1,2,..,6} e^{i\mathbf{k}_j\cdot\mathbf{x}},\quad w = \sum_{j=1,2,..,6} e^{i\mathbf{k}'_j\cdot\mathbf{x}}.$$

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Symmetries

$\mathbf{S}$ represents the imparity symmetry: $\mathbf{S}u = -u$

$$\mathbf{S}\mathbf{L}_0 = \mathbf{L}_0\mathbf{S}, \quad \mathbf{S}u^3 = (\mathbf{S}u)^3.$$

$\mathbf{R}_\theta$ rotation of angle θ, centered at the origin

$$(\mathbf{R}_\theta u)(\mathbf{x}) = u(\mathbf{R}_{-\theta}\mathbf{x}),$$

$$\mathbf{R}_\theta\mathbf{L}_0 = \mathbf{L}_0\mathbf{R}_\theta, \quad \mathbf{R}_\theta u^3 = (\mathbf{R}_\theta u)^3.$$

τ represents the symmetry with respect to the bisectrix of wave vectors $\mathbf{k}_1$ and $\mathbf{k}_1'$.

$$\tau\mathbf{L}_0 = \mathbf{L}_0\tau, \quad \tau u^3 = (\tau u)^3. \tag{2}$$

Then

$$\mathbf{R}_{\pi/3}v = v, \ \mathbf{R}_{\pi/3}w = w, \ \tau v = w, \ \tau w = v$$

Formal series

We look for a formal solution of SHE as

$$u = \sum_{n\geq 1} \varepsilon^n u_n, \;\; \mu = \sum_{n\geq 1} \varepsilon^n \mu_n, \;\; \varepsilon \text{ defined by the choice of } u_1$$

$$\text{order } \varepsilon : \mathbf{L}_0 u_1 = 0, \;\; u_1 \text{ lies in the kernel of } \mathbf{L}_0$$

$$u_1 = w + \beta_1 v$$

the coefficient in front of w fixes the choice of the scale ε, provided that we choose to impose $\langle u_n, w\rangle_0 = 0, \quad n = 2, 3, \ldots$

$$\text{order } \varepsilon^2\text{: } \mathbf{L}_0 u_2 = \mu_1 u_1,$$

and the compatibility condition gives

$$\mu_1 = 0, \;\; u_2 = \beta_2 v.$$

Formal series-continued

$$\text{Order } \varepsilon^3\text{: } \mathbf{L}_0 u_3 = \mu_2 u_1 - u_1^3.$$

$$\begin{aligned} \text{Compatib: } a\mu_2 - c - 3b\beta_1^2 &= 0, \\ a\beta_1\mu_2 - 3b\beta_1 - c\beta_1^3 &= 0, \end{aligned}$$

where $a = 6,\ b = 36,\ c = 90,$
$\langle v^2 w, v\rangle = \langle w^2 v, w\rangle = \langle v^3, w\rangle = \langle w^3, v\rangle = 0.$

$$\text{This gives } (c - 3b)(\beta_1^3 - \beta_1) = 0,$$

$$\mu_2 = \frac{c}{a} + 3\frac{b}{a}\beta_1^2,\ u_3 = \beta_3 v + \widetilde{u}_3,\ \langle \widetilde{u}_3, v\rangle = \langle \widetilde{u}_3, w\rangle = 0.$$

$\widetilde{u}_3$ only contains Fourier modes $e^{i\mathbf{k}\cdot\mathbf{x}}$ with $\mathbf{k} = m_1'\mathbf{k}_1' + m_2'\mathbf{k}_2'$

First case: $\beta_1 = 0$, then $\mu_2 = 15$

Second case: $\beta_1 = \pm 1$, then $\mu_2 = 33$, $\tau u_1 = \beta_1 u_1,\ \tau\widetilde{u}_3 = \beta_1\widetilde{u}_3$

higher orders

For $\beta_1 = 0$ we obtain the classical bifurcating hexagonal-symmetric expansion (u_n is orthogonal to v for all n).
For $\beta_1 = \pm 1$ the expansions are uniquely determined.
$u_1 = w + \beta_1 v, \ \tau u_1 = \beta_1 u_1$

$$\beta_1 = 1 \qquad \text{leads to } \tau u = u,$$
$$\beta_1 = -1 \qquad \text{leads to } \tau u = -u.$$

Formal series (end)

Theorem

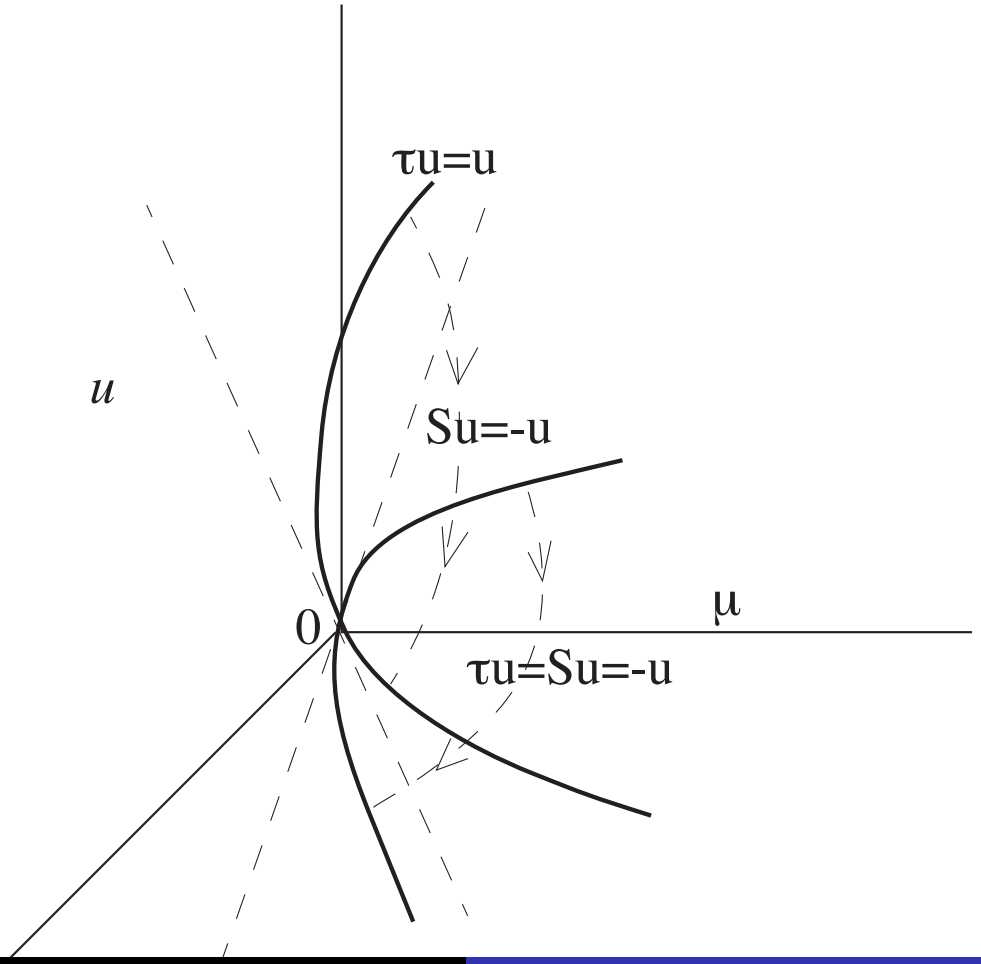
Let us consider the Swift-Hohenberg model PDE . The superposition of two hexagonal patterns, differing by a small rotation of angle $\alpha \in \mathcal{E}_0$, leads to formal expansions in powers of an amplitude ε, of new bifurcating patterns invariant under rotations of angle $\pi/3$. We obtain two new branches of patterns, with formal expansions of the form

$$\begin{aligned} u &= \varepsilon(w + \beta_1 v) + \varepsilon^3 \widetilde{u_3} + \ldots \varepsilon^{2n+1} \widetilde{u_{2n+1}} + .., \quad \beta_1 = \pm 1, \\ \langle \widetilde{u_{2n+1}}, v \rangle &= \langle \widetilde{u_{2n+1}}, w \rangle = 0, \ \tau \widetilde{u_{2n+1}} = \beta_1 \widetilde{u_{2n+1}}, \ \tau u = \beta_1 u, \\ \mu &= \varepsilon^2 \mu_2 + \varepsilon^4 \mu_4 + \ldots + \varepsilon^{2n} \mu_{2n} + .., \ \mu_2 > 0, \end{aligned}$$

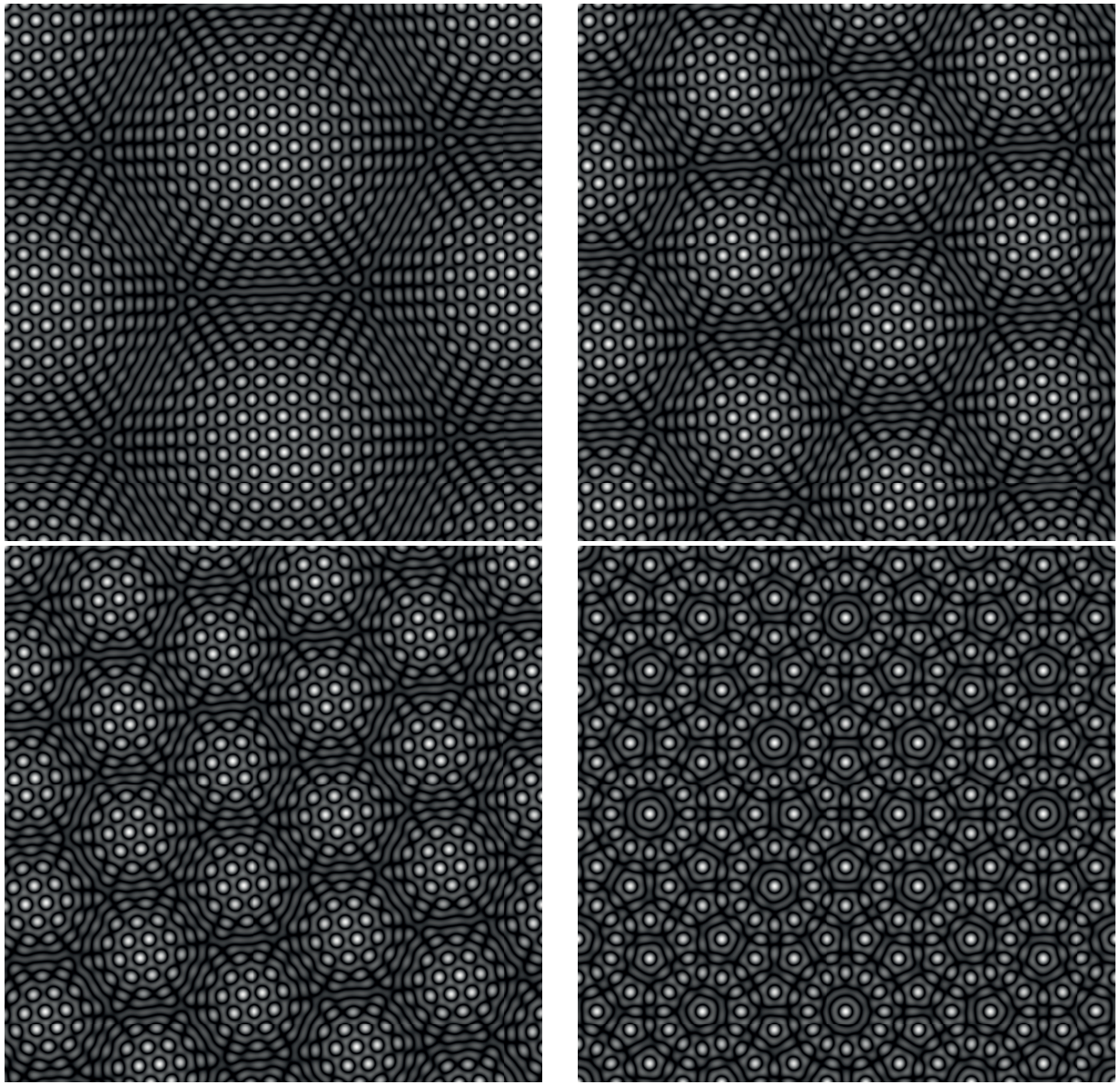
$$v = \sum_{j=1,2,..,6} e^{i\mathbf{k}_j \cdot \mathbf{x}}, \quad w = \sum_{j=1,2,..,6} e^{i\mathbf{k}'_j \cdot \mathbf{x}}, \ (\mathbf{k}_1, \mathbf{k}'_1) = \alpha.$$

For $\varepsilon \in \mathcal{E}_p \cap \mathcal{E}_0$ the expansions converge, giving periodic patterns with hexagonal symmetry.

2 branches of quasipatterns



Solutions of SHE



Superposition solution of SHE for $\alpha = 4^o, 7^o, 10^o, 30^o$

Existence of quasipatterns

Theorem

Assume $\alpha \in \mathcal{E}_2 \cap \mathcal{E}_0 \cap \mathcal{E}_{qp}$ *which is a full measure set, and assume that a transversality condition holds. Then there exist* $s_0 > 2$ *and* $\varepsilon_0 > 0$ *such that for an asymptotically full measure set of values of* $\varepsilon \in (0, \varepsilon_0)$, *there exists a bifurcating quasipattern solution of SHE, invariant under rotation of angle* $\pi/3$, *of the form*

$$\begin{aligned} u &= U_\varepsilon + \varepsilon^4 \widetilde{u}(\varepsilon), \ \widetilde{u} \in \{v, w\}^\perp, \\ U_\varepsilon &= \varepsilon(w + \beta_1 v) + \varepsilon^3 \widetilde{u}_3, \ \beta_1 = \pm 1, \tau u = \beta_1 u, \\ \mu &= \varepsilon^2 \mu_2 + \varepsilon^4 \mu_4 + \widetilde{\mu}(\varepsilon), \end{aligned}$$

where $\widetilde{u}(\varepsilon) \in \mathbf{Q}_0 \mathcal{H}_{s_0}$, $w, v, \widetilde{u}_3, \mu_2, \mu_4$ *are defined above, and functions of* ε *are* C^1 *with* $\widetilde{u}(0) = 0$, $\widetilde{\mu}(\varepsilon) = \mathcal{O}(\varepsilon^6)$. $\mathbf{S}u = -u$ *corresponds to change* ε *into* $-\varepsilon$.

Idea of Proof 1

$$\begin{aligned} u &= U_\varepsilon + \varepsilon^4 W, \quad W = \widetilde{u} + \beta v, \ \widetilde{u} \in \{v, w\}^\perp, \\ U_\varepsilon &= \varepsilon(w + \beta_1 v) + \varepsilon^3 \widetilde{u}_3, \ \beta_1 = \pm 1, \\ \mu &= \mu_\varepsilon + \widetilde{\mu}, \quad \mu_\varepsilon = \varepsilon^2 \mu_2 + \varepsilon^4 \mu_4. \end{aligned}$$

$$(\mathbf{L}_0 - \widetilde{\mu})\widetilde{u} + g(\varepsilon, \beta, \widetilde{\mu}) + \mathcal{B}_{\varepsilon,\beta}\widetilde{u} + \mathcal{C}_{\varepsilon,\beta}(\widetilde{u}) = 0,$$

where $\mathcal{B}_{\varepsilon,\beta}$ is linear and $O(\varepsilon^2)$ in $\mathcal{H}_s, s \geq 0$, and $\mathcal{C}_{\varepsilon,\beta}$ is at least quadratic and $O(\varepsilon^5)$ in $\mathcal{H}_s, s > 2$.
We expect to solve this range-equation with respect to $\widetilde{u}$ which should be of order $O(\varepsilon)$, and put it into

Bifurcation equations:

$$\begin{aligned} a\widetilde{\mu} - 6\varepsilon^5 b\beta_1\beta - 3\varepsilon^5\langle u_1^2\widetilde{u}, w\rangle &= \mathcal{O}(\varepsilon^6), \\ -\beta_1 a\widetilde{\mu} + 2c\varepsilon^5\beta + 3\varepsilon^5\langle u_1^2\widetilde{u}, v\rangle &= \mathcal{O}(\varepsilon^6). \end{aligned}$$

Then we solve with respect to $(\widetilde{\mu}, \beta)$, and find $(\widetilde{\mu}, \beta) = (O(\varepsilon^6), O(\varepsilon))$. Finally $\beta(\varepsilon) \equiv 0$ by a symmetry argument.

Idea of Proof (continued)

We have a small divisor problem:

$$\widetilde{\mathbf{L}_0}^{-1} e^{i\mathbf{k}\cdot\mathbf{x}} = \frac{1}{(|\mathbf{k}|^2-1)^2} e^{i\mathbf{k}\cdot\mathbf{x}}$$

with $(|\mathbf{k}|^2-1)^2 \geq cN_{\mathbf{k}}^{-13}$
Nash-Moser method needs to invert the differential at any V near 0: $\mathcal{L}_{\varepsilon,\beta,V} - \widetilde{\mu}\mathbb{I}$ where $\mathcal{L}_{\varepsilon,\beta,V}$ acts in $\mathbf{Q}_0\mathcal{H}_t,\ \ t \geq 0$ and is defined by

$$\mathcal{L}_{\varepsilon,\beta,V} = \mathbf{L}_0 - \mu_\varepsilon\mathbb{I} + 3\mathbf{Q}_0(U_\varepsilon^2\cdot) - 6\varepsilon^4\mathbf{Q}_0[U_\varepsilon(V+\beta v)(\cdot)] - 3\varepsilon^8\mathbf{Q}_0[(V+\beta v)^2(\cdot)]$$

inverse of $\mathcal{L}_{\varepsilon,\beta,V} - \widetilde{\mu}\mathbb{I}$

Definition

Truncation of the space. Let $s \geq 0$ and $N > 1$ be an integer, we define $E_N := \Pi_N \mathbf{Q}_0 \mathcal{H}_s$, which consists in keeping in the Fourier expansion of $\widetilde{u} \in \mathbf{Q}_0 \mathcal{H}_s$ only those $\mathbf{k} \in \Gamma$ such that $N_{\mathbf{k}} \leq N$. By construction we obtain

$$||(\Pi_N \mathbf{L}_0 \Pi_N)^{-1}||_s \leq c_0 (1+N)^{13}.$$

Inverse of $\mathcal{L}_{\varepsilon,\beta,V} - \widetilde{\mu}\mathbb{I}$ for $N < M_\varepsilon$

Lemma

Let $S > s_0 > 2$ and $\varepsilon_0 > 0$ small enough and $\alpha \in (\mathcal{E}_1 \cap \mathcal{E}_0) \cup \mathcal{E}_{\mathbb{Q}}$. Then for $0 < \varepsilon \leq \varepsilon_0$ and $N \leq M_\varepsilon$ with $M_\varepsilon := \left[\frac{c_1}{\varepsilon^{2/13}}\right]$ and $(\varepsilon, \widetilde{\mu}, \beta, V) \in [-\varepsilon_0, \varepsilon_0] \times [-\varepsilon^2, \varepsilon^2] \times [-\beta_0, \beta_0] \times E_N$, the following holds for $s \in [s_0, S]$ and V such that $||V||_s \leq 1$,
$||(\Pi_N(\mathcal{L}_{\varepsilon,\beta,V} - \widetilde{\mu}\mathbb{I})\Pi_N)^{-1}||_s \leq 2c_0(1+N)^{13}$

Inverse of $\mathcal{L}_{\varepsilon,\beta,V} - \widetilde{\mu}\mathbb{I}$ for large N

define $\Lambda := \{\varepsilon, \tilde{\mu}); \varepsilon \in [-\varepsilon_0, \varepsilon_0], \tilde{\mu} \in [-\varepsilon^2, \varepsilon^2]\}$, and for $M > 0$, $s_0 > 2$,

$$\begin{aligned} \mathcal{U}_M^{(N)} \quad : \quad &= \left\{ V \in C^1(\Lambda \times [-\beta_0, \beta_0], E_N); V(0, \widetilde{\mu}, \beta) = 0, \right. \\ &\left. ||V||_{s_0} \leq 1, ||\partial_{\varepsilon,\beta} V||_{s_0} \leq M, ||\partial_{\widetilde{\mu}} V||_{s_0} \leq (M/\varepsilon^2) \right\}. \end{aligned}$$

For $V \in \mathcal{U}_M^{(N)}$, we consider the operator

$$\begin{aligned} \Pi_N(\mathcal{L}_{\varepsilon,\beta,V(\varepsilon,\widetilde{\mu},\beta)} - \widetilde{\mu}\mathbb{I})\Pi_N \quad = \quad & \Pi_N \mathbf{L}_0 \Pi_N - \widetilde{\mu}\mathbb{I}_N + \varepsilon^2 \mathcal{B}_1^{(N)}(\varepsilon) + \\ & + \varepsilon^5 \mathcal{B}_2^{(N)}(\varepsilon, \beta, V(\varepsilon, \widetilde{\mu}, \beta)), \end{aligned}$$

$\Pi_N \mathbf{L}_0 \Pi_N$, $\mathcal{B}_1^{(N)}$, $\mathcal{B}_2^{(N)}$ selfadjoint in $\Pi_N \mathbf{Q}_0 \mathcal{H}_0$.and analytic in their arguments.

Eigenvalues of $\Pi_N(\mathcal{L}_{\varepsilon,\beta,V(\varepsilon,\widetilde{\mu},\beta)} - \widetilde{\mu}\mathbb{I})\Pi_N$ have the form

$$\sigma_j(\varepsilon, \widetilde{\mu}, \beta) = f_j(\varepsilon, \widetilde{\mu}, \beta) - \widetilde{\mu},$$

where f_j is Lipschitz in $(\varepsilon, \widetilde{\mu}, \beta)$ and $|\frac{\partial f_j}{\partial \widetilde{\mu}}| \leq c\varepsilon^3$

Inverse of $\mathcal{L}_{\varepsilon,\beta,V} - \widetilde{\mu}\mathbb{I}$ for large N (continued 1)

Bad set of $\tilde{\mu}$

$$B^{(N)}_{\varepsilon,\beta,\gamma}(V) = \Big\{\widetilde{\mu} \in [-\varepsilon_0, \varepsilon_0]; (\varepsilon, \beta, V) \in [-\varepsilon_0, \varepsilon_0] \times [-\beta_0, \beta_0] \times \mathcal{U}^{(N)}_M, \\ \exists j \in \{1, ...\mathcal{N}\}, |\sigma_j(\varepsilon, \widetilde{\mu}, \beta)| < \frac{\gamma}{N^\tau}\Big\}$$

$$B^{(N)}_{\varepsilon,\beta,\gamma}(V) = \cup_{j=1}^{\mathcal{N}} (\widetilde{\mu}_j^-(\varepsilon, \beta), \widetilde{\mu}_j^+(\varepsilon, \beta)),$$

$$0 < \widetilde{\mu}_j^+(\varepsilon, \beta) - \widetilde{\mu}_j^-(\varepsilon, \beta) \leq \frac{4\gamma}{N^\tau},$$

$$\mathrm{meas}(B^{(N)}_{\varepsilon,\beta,\gamma}(V)) \leq \frac{4b\gamma}{N^{\tau-4}},$$

$\widetilde{\mu}_j^{\pm}(\varepsilon, \beta)$ are Lipschitz continuous with a small Lip constant.

Good set of $\tilde{\mu}$: $G^{(N)}_{\varepsilon,\beta,\gamma}(V) := [-\varepsilon_0, \varepsilon_0] \backslash B^{(N)}_{\varepsilon,\beta,\gamma}(V)$.

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Inverse of $\mathcal{L}_{\varepsilon,\beta,V} - \widetilde{\mu}\mathbb{I}$ for large N (continued 2)

Lemma

Assume $\gamma \leq \widetilde{\gamma} = 2^{13/2+1}c_0)$ *and* $\tau > 46$. *For* $V \in \mathcal{U}_M^{(N)}$ *and* $(\varepsilon, \beta) \in [-\varepsilon_0, \varepsilon_0] \times [-\beta_0, \beta_0]$ *fixed, then if* $\widetilde{\mu} \in G^{(N)}_{\varepsilon,\beta,\gamma}(V) \cap [-\varepsilon^2, \varepsilon^2]$, $N > 1$

$$||(\Pi_N(\mathcal{L}_{\varepsilon,\beta,V(\varepsilon,\widetilde{\mu},\beta)} - \widetilde{\mu}\mathbb{I})\Pi_N)^{-1}||_0 \leq \frac{N^\tau}{\gamma}.$$

Moreover, for $N > M_\varepsilon$, *the measure of the "bad set"* $B^{(N)}_{\varepsilon,\beta,\gamma}(V)$ *is bounded by* $4b\gamma/N^{\tau-4}$, *while it is* 0 *for* $N \leq M_\varepsilon$.

This estimate is in $\mathcal{L}(\mathbf{Q}_0\mathcal{H}_0)$. In fact, we need to obtain a tame estimate for $(\Pi_N(\mathcal{L}_{\varepsilon,\beta,V(\varepsilon,\widetilde{\mu},\beta)} - \widetilde{\mu}\mathbb{I})\Pi_N)^{-1}$ in $\mathcal{L}(\mathbf{Q}_0\mathcal{H}_s)$ for $s > 0$, with an exponent on N not depending on s.
We use Bourgain 1995, Craig 2000, Berti-Bolle 2010 with a suitable adaptation.

Resolution of the Range equation

We set $\tilde{\mu} = \varepsilon^2 \widehat{\mu}$
Nash-Moser method, following Berti-Bolle-Procesi 2010 leads to:

Theorem

Assume $\alpha \in \mathcal{E}_2 \cap \mathcal{E}_0 \cap \mathcal{E}_{qp}$ and let s_0 be as in Lemma above. Then for all $0 < \gamma \leq \widetilde{\gamma}$ there exist $\varepsilon_2(\gamma) \in (0, \varepsilon_0)$ and a C^1- map V : $(0, \varepsilon_2(\gamma)) \times [-1, 1] \to \mathcal{H}_{s_0+4}$ such that $V(0, \widehat{\mu}, \beta) = 0$ and if $\varepsilon \in (0, \varepsilon_2(\gamma))$, $\widehat{\mu} \in [-1, 1] \backslash C_{\varepsilon,\beta,\gamma}$, the function $\widetilde{u} = V(\varepsilon, \widehat{\mu}, \beta)$ is solution of the range equation. Here $C_{\varepsilon,\beta,\gamma}$ is a subset of $[-1, 1]$, which is a Lipschitz function of (ε, β) and has a Lebesgue measure less than $C\gamma|\varepsilon|^3$ for some constant $C > 0$, independent of $(\varepsilon, \beta, \gamma)$.

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Resolution of the Bifurcation equations

$$a\widehat{\mu} - 6\varepsilon^3 b\beta_1\beta - 3\varepsilon^3\langle u_1^2 V(\varepsilon,\widehat{\mu},\beta), w\rangle = \mathcal{O}(\varepsilon^4),$$
$$-\beta_1 a\widehat{\mu} + 2c\varepsilon^3\beta + 3\varepsilon^3\langle u_1^2 V(\varepsilon,\widehat{\mu},\beta), v\rangle = \mathcal{O}(\varepsilon^4).$$

Implicit function theorem gives

$$\widetilde{\mu} = \varepsilon^2\widehat{\mu} = \varepsilon^6 h(\varepsilon),\ \ \beta = \varepsilon g(\varepsilon), \text{ (H)}$$

$\varepsilon h(\varepsilon)$ and $\varepsilon g(\varepsilon)$ are C^1 functions of $\varepsilon \in [0, \varepsilon_1]$.

Define "bad layers" of degree N: $BS_N(V) := \{(\varepsilon, \widetilde{\mu}, \beta) \in \Lambda \times [-\beta_0, \beta_0]; \exists j; \widetilde{\mu} \in (\widetilde{\mu}_j^-(\varepsilon,\beta), \widetilde{\mu}_j^+(\varepsilon,\beta))\}$.
In the 3-dimensional space $(\varepsilon, \widetilde{\mu}, \beta)$ we need to check that the curve (H) crosses transversally the bad set formed by the infinitely many thin layers $\cup_{n\in\mathbb{N}} B_n S_{N_n}(V_{n-1})$, where $N_n = [N_0(\gamma)]^{2^n}$, and V_n are the successive points obtained in the Newton iteration of the Nash-Moser method.

Transversality condition

The intersection of the surface $\beta = \varepsilon g(\varepsilon)$ with $\cup_{n\in\mathbb{N}} B_n S_{N_n}(V_{n-1})$ is a set of bad strips, Lipschitz continuous in ε.

Transversality condition: *Let $\widetilde{\mu}(\varepsilon)$ be any one of the limiting curves of the bad strips given by $\{\beta = \varepsilon g(\varepsilon)\} \cap_{n\in\mathbb{N}} BS_{N_n}(V_{n-1})$. Then we assume that for any of these curves, there exists $c > 0$ independent of N, such that for $h \in \mathbb{R}$ in a neighborhood of 0, the following inequality holds:*

$$|\widetilde{\mu}(\varepsilon + h) - \widetilde{\mu}(\varepsilon)| \geq c|\varepsilon|^3|h|. \tag{3}$$

NB: when eigenvalues σ_j are not multiple, $\widetilde{\mu}^{\pm}(\varepsilon, \beta(\varepsilon)$ has a slope $|t(\varepsilon)| > c_2|\varepsilon|$. So, the transversality condition is very weak. Moreover notice the slope of (H) is of order ε^5.

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final result 1

measure of "bad" $\widetilde{\mu}$'s $< C\gamma|\varepsilon|^5$ hence
measure of "bad" ε's $< \frac{C\gamma|\varepsilon|^5}{min|t|} < \frac{C\gamma|\varepsilon|^5}{c|\varepsilon|^3} \leq \frac{C\gamma\varepsilon^2}{c}$.

The complementary subset in $(0, \varepsilon_2)$, is the good set of $|\varepsilon|$, which is of asymptotic full measure since $[|\varepsilon| - \frac{C\gamma\varepsilon^2}{c}]/|\varepsilon| \to 1$ as $\varepsilon \to 0$.

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final result 2: $\beta(\varepsilon) \equiv 0$

Use a symmetry argument
$\tau U_\varepsilon = \beta_1 U_\varepsilon$,
For $\beta_1 = 1$,

$$\tau(\widetilde{u}(\varepsilon) + \beta(\varepsilon)v) = \tau\widetilde{u}(\varepsilon) + \beta(\varepsilon)w = \widetilde{u}(\varepsilon) + \beta(\varepsilon)v$$

by the uniqueness of the solution u. Hence, $\beta(\varepsilon) \equiv 0$.

For $\beta_1 = -1$,

$$\tau\widetilde{u}(\varepsilon) + \beta(\varepsilon)w = -\widetilde{u}(\varepsilon) - \beta(\varepsilon)v$$

by the uniqueness of the solution $-u$. This implies that in all cases

$$\tau\widetilde{u}(\varepsilon) = \beta_1\widetilde{u}(\varepsilon), \ \beta(\varepsilon) \equiv 0$$

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Existence of quasipatterns

Theorem

Assume $\alpha \in \mathcal{E}_2 \cap \mathcal{E}_0 \cap \mathcal{E}_{qp}$ *which is a full measure set, and assume that a transversality condition holds. Then there exist* $s_0 > 2$ *and* $\varepsilon_0 > 0$ *such that for an asymptotically full measure set of values of* $\varepsilon \in (0, \varepsilon_0)$, *there exist two branches of bifurcating quasipattern solutions of SHE, invariant under rotation of angle* $\pi/3$, *of the form*

$$\begin{aligned} u &= U_\varepsilon + \varepsilon^4 \widetilde{u}(\varepsilon), \ \widetilde{u} \in \{v, w\}^\perp, \\ U_\varepsilon &= \varepsilon(w + \beta_1 v) + \varepsilon^3 \widetilde{u}_3, \ \beta_1 = \pm 1, \tau u = \beta_1 u, \\ \mu &= \varepsilon^2 \mu_2 + \varepsilon^4 \mu_4 + \widetilde{\mu}(\varepsilon), \end{aligned}$$

where $\widetilde{u}(\varepsilon) \in \mathbf{Q}_0 \mathcal{H}_{s_0}$, $w, v, \widetilde{u}_3, \mu_2, \mu_4$ *are defined above, and functions of* ε *are* C^1 *with* $\widetilde{u}(0) = 0$, $\widetilde{\mu}(\varepsilon) = \mathcal{O}(\varepsilon^6)$. $\mathbf{S}u = -u$ *corresponds to change* ε *into* $-\varepsilon$.

References on Quasipatterns

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