Quasipatterns in the superposition of two hexagonal patterns for the Swift-Hohenberg PDE

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Superposition experiments

Superposition of two hexagonal patterns: $0^\circ, 3^\circ, 5^\circ, 10^\circ, 20^\circ, 30^\circ$
Definition of the quasilattice

\[ \Gamma = \{ k \in \mathbb{R}^2; k = \sum_{j=1,...,6} m_j k_j + m'_j k'_j, \ m_j, m'_j \in \mathbb{N} \} \]

Special angles \( \mathcal{E}_p := \{ \alpha \in \mathbb{R}/2\pi\mathbb{Z}; \cos \alpha \in \mathbb{Q}, \cos(\alpha + \pi/3) \in \mathbb{Q} \} \).

Lemma

The set \( \mathcal{E}_p \) has a zero measure in \( \mathbb{R}/2\pi\mathbb{Z} \).

(i) If the wave vectors \( k_1, k_2, k'_1, k'_2 \) are not independent on \( \mathbb{Q} \), then \( \alpha \in \mathcal{E}_p \).

(ii) If \( \alpha \in \mathcal{E}_p \) then the lattice \( \Gamma \) is periodic with an hexagonal symmetry, and wave vectors \( k_1, k_2, k'_1, k'_2 \) are combinations of only two smaller vectors, of equal length making an angle \( 2\pi/3 \).
Fourier series

$u(x, y)$ function under the form of a Fourier expansion

$$u = \sum_{k \in \Gamma} u^{(k)} e^{i k \cdot x}, \quad u^{(k)} = \overline{u}^{(-k)} \in \mathbb{C}. \quad (1)$$

$k \in \Gamma$ may be written as

$$k = x_1 k_1 + x_2 k_2 + x_3 k'_1 + x_4 k'_2, \quad (x_1, x_2, x_3, x_4) \in \mathbb{Z}^4,$$

If $\alpha \in \mathcal{E}_{qp} = \mathcal{E}_p^c$, $\Gamma$ spans a 4-dimensional vector space on $\mathbb{Q}$.

$$N_k = \sum_{j=1,...,4} |x_j| \text{ is a norm of } k$$

Hilbert spaces $\mathcal{H}_s$, $s \geq 0$:

$$\mathcal{H}_s = \left\{ u = \sum_{k \in \Gamma} u^{(k)} e^{i k \cdot x}, \quad u^{(k)} = \overline{u}^{(-k)} \in \mathbb{C}, \quad \sum_{k \in \Gamma} |u^{(k)}|^2 (1 + N_k^2)^s < \infty \right\},$$
basic Lemmas

Lemma

If \( \alpha \in \mathcal{E}_{qp} \), a function defined by a convergent Fourier series as above represents a quasipattern, i.e. is quasiperiodic in all directions.

Lemma

For nearly all \( \alpha \in (0, \pi/6) \), in particular for \( \alpha \in \mathbb{Q}\pi \cap (0, \pi/6] \), the only solutions of \( |k(x)| = 1 \) are \( \pm k_j, \pm k'_j \) \( j = 1, 2 \) and \( k = \pm k_3 \), or \( \pm k'_3 \), i.e. corresponding to \( x = (\pm 1, \mp 1, 0, 0) \) or \( (0, 0, \pm 1, \mp 1) \).

\( \mathcal{E}_0 \) is the set of \( \alpha \)'s such that Lemma above applies.

Lemma

For nearly all \( \alpha \in \mathcal{E}_{qp} \cap (0, \pi/6) \), and for \( \varepsilon > 0 \), there exists \( c > 0 \) such that, for all \( |x| > 0 \) such that \( |k(x)| \neq 1 \),
\[
(|k(x)|^2 - 1)^2 \geq \frac{\varepsilon}{|x|^{12+\varepsilon}}
\]
holds.
Steady Swift-Hohenberg equation in $\mathbb{R}^2$

$$(1 + \Delta)^2 u = \mu u - u^3, \ x \in \mathbb{R}^2 \rightarrow u(x) \in \mathbb{R}$$

$$e^{ik \cdot x} \in \text{Ker}\{(1 + \Delta)^2 - \mu\} \text{ in } \mathcal{H}_s$$

iff Dispersion equation holds: $(1 - |k|^2)^2 = \mu, \ k \in \Gamma$

For $\mu = 0$ all wave vectors $k$ with $|k| = 1$ are critical

We choose to look for solutions in $\mathcal{H}_s$, for $\alpha \in \mathcal{E}_{qp} \cap \mathcal{E}_0$, i.e. quasiperiodic in $\mathbb{R}^2$, moreover invariant under rotations of angle $\pi/3$ and bifurcating for $\mu$ close to 0.

define $L_0 = (1 + \Delta)^2$

For $\alpha \in \mathcal{E}_0$ Ker$L_0$ is 2-dimensional spanned by

$$v = \sum_{j=1,2,\ldots,6} e^{ik_j \cdot x}, \ w = \sum_{j=1,2,\ldots,6} e^{ik'_j \cdot x}.$$
**Symmetries**

\( S \) represents the imparity symmetry: \( Su = -u \)

\[
SL_0 = L_0S, \quad Su^3 = (Su)^3.
\]

\( R_{\theta} \) rotation of angle \( \theta \), centered at the origin

\[
(R_{\theta}u)(x) = u(R_{-\theta}x),
\]

\[
R_{\theta}L_0 = L_0R_{\theta}, \quad R_{\theta}u^3 = (R_{\theta}u)^3.
\]

\( \tau \) represents the symmetry with respect to the bisectrix of wave vectors \( k_1 \) and \( k'_1 \).

\[
\tau L_0 = L_0\tau, \quad \tau u^3 = (\tau u)^3. \tag{2}
\]

Then

\[
R_{\pi/3}v = v, \quad R_{\pi/3}w = w, \quad \tau v = w, \quad \tau w = v
\]
We look for a formal solution of SHE as

\[ u = \sum_{n \geq 1} \varepsilon^n u_n, \quad \mu = \sum_{n \geq 1} \varepsilon^n \mu_n, \quad \varepsilon \text{ defined by the choice of } u_1 \]

order \( \varepsilon \): \( L_0 u_1 = 0 \), \( u_1 \) lies in the kernel of \( L_0 \)

\[ u_1 = w + \beta_1 \nu \]

the coefficient in front of \( w \) fixes the choice of the scale \( \varepsilon \), provided that we choose to impose \( \langle u_n, w \rangle_0 = 0, \quad n = 2, 3, \ldots \)

order \( \varepsilon^2 \): \( L_0 u_2 = \mu_1 u_1 \), and the compatibility condition gives

\[ \mu_1 = 0, \quad u_2 = \beta_2 \nu. \]
Order $\varepsilon^3$: $L_0 u_3 = \mu_2 u_1 - u_1^3$.

Compatib: 
\[
\begin{align*}
 a\mu_2 - c - 3b\beta_1^2 &= 0, \\
 a\beta_1\mu_2 - 3b\beta_1 - c\beta_1^3 &= 0,
\end{align*}
\]

where $a = 6$, $b = 36$, $c = 90$,
\[
\langle v^2 w, v \rangle = \langle w^2 v, w \rangle = \langle v^3, w \rangle = \langle w^3, v \rangle = 0.
\]

This gives $(c - 3b)(\beta_1^3 - \beta_1) = 0$,

\[
\mu_2 = \frac{c}{a} + 3\frac{b}{a}\beta_1^2, \quad u_3 = \beta_3 v + \tilde{u}_3, \quad \langle \tilde{u}_3, v \rangle = \langle \tilde{u}_3, w \rangle = 0.
\]

$\tilde{u}_3$ only contains Fourier modes $e^{ik \cdot x}$ with $k = m'_1 k'_1 + m'_2 k'_2$

First case: $\beta_1 = 0$, then $\mu_2 = 15$
Second case: $\beta_1 = \pm 1$, then $\mu_2 = 33$, $\tau u_1 = \beta_1 u_1$, $\tau \tilde{u}_3 = \beta_1 \tilde{u}_3$
For $\beta_1 = 0$ we obtain the classical bifurcating hexagonal-symmetric expansion ($u_n$ is orthogonal to $v$ for all $n$).

For $\beta_1 = \pm 1$ the expansions are uniquely determined.

$u_1 = w + \beta_1 v$, $\tau u_1 = \beta_1 u_1$

$\beta_1 = 1$ leads to $\tau u = u$,

$\beta_1 = -1$ leads to $\tau u = -u$. 
Theorem

Let us consider the Swift-Hohenberg model PDE. The superposition of two hexagonal patterns, differing by a small rotation of angle $\alpha \in \mathcal{E}_0$, leads to formal expansions in powers of an amplitude $\varepsilon$, of new bifurcating patterns invariant under rotations of angle $\pi/3$. We obtain two new branches of patterns, with formal expansions of the form

$$ u = \varepsilon(w + \beta_1 v) + \varepsilon^3 \tilde{u}_3 + \ldots \varepsilon^{2n+1} \tilde{u}_{2n+1} + \ldots, \quad \beta_1 = \pm 1, $$

$$ \langle \tilde{u}_{2n+1}, v \rangle = \langle \tilde{u}_{2n+1}, w \rangle = 0, \quad \tau \tilde{u}_{2n+1} = \beta_1 \tilde{u}_{2n+1}, \quad \tau u = \beta_1 u, $$

$$ \mu = \varepsilon^2 \mu_2 + \varepsilon^4 \mu_4 + \ldots + \varepsilon^{2n} \mu_{2n} + \ldots, \quad \mu_2 > 0, $$

$$ v = \sum_{j=1,2,\ldots,6} e^{ik_j \cdot x}, \quad w = \sum_{j=1,2,\ldots,6} e^{ik'_j \cdot x}, \quad \langle k_1, k'_1 \rangle = \alpha. $$

For $\varepsilon \in \mathcal{E}_p \cap \mathcal{E}_0$ the expansions converge, giving periodic patterns with hexagonal symmetry.
2 branches of quasipatterns

\[ u = \mu \]

\[ \tau u = u \quad \text{and} \quad Su = -u \]

\[ \tau u = Su = -u \]
Superposition solution of SHE for $\alpha = 4^\circ, 7^\circ, 10^\circ, 30^\circ$
Existence of quasipatterns

**Theorem**

Assume \( \alpha \in E_2 \cap E_0 \cap E_{qp} \) which is a full measure set, and assume that a transversality condition holds. Then there exist \( s_0 > 2 \) and \( \varepsilon_0 > 0 \) such that for an asymptotically full measure set of values of \( \varepsilon \in (0, \varepsilon_0) \), there exists a bifurcating quasipattern solution of SHE, invariant under rotation of angle \( \pi/3 \), of the form

\[
\begin{align*}
    u &= U_\varepsilon + \varepsilon^4 \tilde{u}(\varepsilon), \quad \tilde{u} \in \{v, w\}, \\
    U_\varepsilon &= \varepsilon(w + \beta_1 v) + \varepsilon^3 \tilde{u}_3, \quad \beta_1 = \pm 1, \tau u = \beta_1 u, \\
    \mu &= \varepsilon^2 \mu_2 + \varepsilon^4 \mu_4 + \tilde{\mu}(\varepsilon),
\end{align*}
\]

where \( \tilde{u}(\varepsilon) \in Q_0 H_{s_0} \), \( w, v, \tilde{u}_3, \mu_2, \mu_4 \) are defined above, and functions of \( \varepsilon \) are \( C^1 \) with \( \tilde{u}(0) = 0 \), \( \tilde{\mu}(\varepsilon) = O(\varepsilon^6) \). \( Su = -u \) corresponds to change \( \varepsilon \) into \( -\varepsilon \).
Idea of Proof 1

\[ u = U_\varepsilon + \varepsilon^4 W, \quad W = \tilde{u} + \beta v, \quad \tilde{u} \in \{ v, w \}^\perp, \]
\[ U_\varepsilon = \varepsilon(w + \beta_1 v) + \varepsilon^3 \tilde{u}_3, \quad \beta_1 = \pm 1, \]
\[ \mu = \mu_\varepsilon + \tilde{\mu}, \quad \mu_\varepsilon = \varepsilon^2 \mu_2 + \varepsilon^4 \mu_4. \]

\[(L_0 - \tilde{\mu})\tilde{u} + g(\varepsilon, \beta, \tilde{\mu}) + B_{\varepsilon, \beta} \tilde{u} + C_{\varepsilon, \beta}(\tilde{u}) = 0,\]

where \( B_{\varepsilon, \beta} \) is linear and \( O(\varepsilon^2) \) in \( H_s, s \geq 0 \), and \( C_{\varepsilon, \beta} \) is at least quadratic and \( O(\varepsilon^5) \) in \( H_s, s > 2 \).

We expect to solve this range-equation with respect to \( \tilde{u} \) which should be of order \( O(\varepsilon) \), and put it into

Bifurcation equations:

\[ a\tilde{\mu} - 6\varepsilon^5 b\beta_1 \beta - 3\varepsilon^5 \langle u_1^2 \tilde{u}, w \rangle = O(\varepsilon^6), \]
\[ -\beta_1 a\tilde{\mu} + 2c\varepsilon^5 \beta + 3\varepsilon^5 \langle u_1^2 \tilde{u}, v \rangle = O(\varepsilon^6). \]

Then we solve with respect to \((\tilde{\mu}, \beta)\), and find
\((\tilde{\mu}, \beta) = (O(\varepsilon^6), O(\varepsilon))\). Finally \( \beta(\varepsilon) \equiv 0 \) by a symmetry argument.
We have a small divisor problem:

\[ \tilde{L}_0^{-1} e^{ik \cdot x} = \frac{1}{(|k|^2 - 1)^2} e^{ik \cdot x} \]

with \((|k|^2 - 1)^2 \geq cN_k^{-13}\)

Nash-Moser method needs to invert the differential at any \(V\) near \(0: \mathcal{L}_{\varepsilon, \beta, V} - \tilde{\mu} \mathbb{I}\) where \(\mathcal{L}_{\varepsilon, \beta, V}\) acts in \(Q_0 \mathcal{H}_t, \ t \geq 0\) and is defined by

\[
\mathcal{L}_{\varepsilon, \beta, V} = L_0 - \mu_{\varepsilon} \mathbb{I} + 3Q_0(U_{\varepsilon}^2 \cdot) - 6\varepsilon^4 Q_0[U_{\varepsilon}(V + \beta v)(\cdot)] - 3\varepsilon^8 Q_0[(V + \beta v)^2(\cdot)]
\]
Definition

Truncation of the space. Let \( s \geq 0 \) and \( N > 1 \) be an integer, we define \( E_N := \prod_N Q_0 \mathcal{H}_s \), which consists in keeping in the Fourier expansion of \( \tilde{u} \in Q_0 \mathcal{H}_s \) only those \( k \in \Gamma \) such that \( N_k \leq N \). By construction we obtain

\[
\| (\prod_N L_0 \prod_N)^{-1} \|_s \leq c_0 (1 + N)^{13}.
\]

Inverse of \( L_{\varepsilon, \beta, V} - \tilde{\mu} \mathbb{I} \) for \( N < M_\varepsilon \)

Lemma

Let \( S > s_0 > 2 \) and \( \varepsilon_0 > 0 \) small enough and \( \alpha \in (E_1 \cap E_0) \cup E_Q \). Then for \( 0 < \varepsilon \leq \varepsilon_0 \) and \( N \leq M_\varepsilon \) with \( M_\varepsilon := \left[ \frac{c_1}{\varepsilon^{2/13}} \right] \) and \((\varepsilon, \tilde{\mu}, \beta, V) \in \left[ -\varepsilon_0, \varepsilon_0 \right] \times \left[ -\varepsilon^2, \varepsilon^2 \right] \times \left[ -\beta_0, \beta_0 \right] \times E_N \), the following holds for \( s \in [s_0, S] \) and \( V \) such that \( \| V \|_s \leq 1 \),

\[
\| (\prod_N (L_{\varepsilon, \beta, V} - \tilde{\mu} \mathbb{I}) \prod_N)^{-1} \|_s \leq 2c_0 (1 + N)^{13}
\]
Inverse of $\mathcal{L}_{\varepsilon, \beta, V} - \tilde{\mu} \mathbb{I}$ for large $N$

define $\Lambda := \{\varepsilon, \tilde{\mu}\}; \varepsilon \in [-\varepsilon_0, \varepsilon_0], \tilde{\mu} \in [-\varepsilon^2, \varepsilon^2]\), and for $M > 0$, $s_0 > 2$,

$$U_M^{(N)} := \{V \in C^1(\Lambda \times [-\beta_0, \beta_0], E_N); V(0, \tilde{\mu}, \beta) = 0, ||V||_{s_0} \leq 1, ||\partial_{\varepsilon, \beta} V||_{s_0} \leq M, ||\partial_{\tilde{\mu}} V||_{s_0} \leq (M/\varepsilon^2)\}.$$

For $V \in U_M^{(N)}$, we consider the operator

$$\Pi_N(\mathcal{L}_{\varepsilon, \beta, V(\varepsilon, \tilde{\mu}, \beta)} - \tilde{\mu} \mathbb{I})\Pi_N = \Pi_N L_0 \Pi_N - \tilde{\mu} \mathbb{I}_N + \varepsilon^2 B_{1}^{(N)}(\varepsilon) + \varepsilon^5 B_{2}^{(N)}(\varepsilon, \beta, V(\varepsilon, \tilde{\mu}, \beta)),$$

$\Pi_N L_0 \Pi_N, B_{1}^{(N)}, B_{2}^{(N)}$ selfadjoint in $\Pi_N Q_0 H_0$. and analytic in their arguments.

Eigenvalues of $\Pi_N(\mathcal{L}_{\varepsilon, \beta, V(\varepsilon, \tilde{\mu}, \beta)} - \tilde{\mu} \mathbb{I})\Pi_N$ have the form

$$\sigma_j(\varepsilon, \tilde{\mu}, \beta) = f_j(\varepsilon, \tilde{\mu}, \beta) - \tilde{\mu},$$

where $f_j$ is Lipschitz in $(\varepsilon, \tilde{\mu}, \beta)$ and $|\frac{\partial f_j}{\partial \tilde{\mu}}| \leq c\varepsilon^3$.

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Inverse of $\mathcal{L}_{\varepsilon, \beta, V} - \tilde{\mu}\mathbb{I}$ for large $N$ (continued 1)

Bad set of $\tilde{\mu}$

$$B^{(N)}_{\varepsilon, \beta, \gamma}(V) = \left\{ \tilde{\mu} \in [-\varepsilon_0, \varepsilon_0]; (\varepsilon, \beta, V) \in [-\varepsilon_0, \varepsilon_0] \times [-\beta_0, \beta_0] \times \mathcal{U}_{M}^{(N)}, \right.$$  
$$\exists j \in \{1, \ldots, N\}, |\sigma_j(\varepsilon, \tilde{\mu}, \beta)| < \frac{\gamma}{N^\tau} \right\}$$

$$B^{(N)}_{\varepsilon, \beta, \gamma}(V) = \bigcup_{j=1}^{N} (\tilde{\mu}_j^-(\varepsilon, \beta), \tilde{\mu}_j^+(\varepsilon, \beta)),$$

$$0 < \tilde{\mu}_j^+(\varepsilon, \beta) - \tilde{\mu}_j^-(\varepsilon, \beta) \leq \frac{4\gamma}{N^\tau},$$

$$\text{meas}(B^{(N)}_{\varepsilon, \beta, \gamma}(V)) \leq \frac{4b\gamma}{N^\tau - 4},$$

$\tilde{\mu}_j^\pm(\varepsilon, \beta)$ are Lipschitz continuous with a small Lip constant.

Good set of $\tilde{\mu}$: $G^{(N)}_{\varepsilon, \beta, \gamma}(V) := [-\varepsilon_0, \varepsilon_0] \setminus B^{(N)}_{\varepsilon, \beta, \gamma}(V)$. 

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Inverse of \( \mathcal{L}_{\varepsilon, \beta, V} - \tilde{\mu} \mathbb{I} \) for large \( N \) (continued 2)

**Lemma**

Assume \( \gamma \leq \tilde{\gamma} = 2^{13/2+1} c_0 \) and \( \tau > 46 \). For \( V \in \mathcal{U}^{(N)}_M \) and \( (\varepsilon, \beta) \in [-\varepsilon_0, \varepsilon_0] \times [-\beta_0, \beta_0] \) fixed, then if \( \tilde{\mu} \in G^{(N)}_{\varepsilon, \beta, \gamma}(V) \cap [-\varepsilon^2, \varepsilon^2] \), \( N > 1 \)

\[
\| (\prod_N (\mathcal{L}_{\varepsilon, \beta, V(\varepsilon, \tilde{\mu}, \beta)} - \tilde{\mu} \mathbb{I}) \prod_N )^{-1} \|_0 \leq \frac{N^\tau}{\gamma}.
\]

Moreover, for \( N > M_\varepsilon \), the measure of the "bad set" \( B^{(N)}_{\varepsilon, \beta, \gamma}(V) \) is bounded by \( 4b \gamma / N^{\tau-4} \), while it is 0 for \( N \leq M_\varepsilon \).

This estimate is in \( \mathcal{L}(Q_0 H_0) \). In fact, we need to obtain a tame estimate for \( (\prod_N (\mathcal{L}_{\varepsilon, \beta, V(\varepsilon, \tilde{\mu}, \beta)} - \tilde{\mu} \mathbb{I}) \prod_N )^{-1} \) in \( \mathcal{L}(Q_0 H_s) \) for \( s > 0 \), with an exponent on \( N \) not depending on \( s \).

We use Bourgain 1995, Craig 2000, Berti-Bolle 2010 with a suitable adaptation.
We set $\tilde{\mu} = \varepsilon^2 \hat{\mu}$

Nash-Moser method, following Berti-Bolle-Procesi 2010 leads to:

**Theorem**

Assume $\alpha \in \mathcal{E}_2 \cap \mathcal{E}_0 \cap \mathcal{E}_{qp}$ and let $s_0$ be as in Lemma above. Then for all $0 < \gamma \leq \tilde{\gamma}$ there exist $\varepsilon_2(\gamma) \in (0, \varepsilon_0)$ and a $C^1-$ map $V : (0, \varepsilon_2(\gamma)) \times [-1, 1] \to \mathcal{H}_{s_0+4}$ such that $V(0, \hat{\mu}, \beta) = 0$ and if $\varepsilon \in (0, \varepsilon_2(\gamma))$, $\hat{\mu} \in [-1, 1] \setminus C_{\varepsilon, \beta, \gamma}$, the function $\tilde{u} = V(\varepsilon, \hat{\mu}, \beta)$ is solution of the range equation. Here $C_{\varepsilon, \beta, \gamma}$ is a subset of $[-1, 1]$, which is a Lipschitz function of $(\varepsilon, \beta)$ and has a Lebesgue measure less than $C\gamma|\varepsilon|^3$ for some constant $C > 0$, independent of $(\varepsilon, \beta, \gamma)$. 

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Resolution of the Bifurcation equations

\begin{align*}
    a\hat{\mu} - 6\varepsilon^3 b\beta_1\beta - 3\varepsilon^3 \langle u_1^2 V(\varepsilon, \hat{\mu}, \beta), w \rangle &= \mathcal{O}(\varepsilon^4), \\
    -\beta_1 a\hat{\mu} + 2c\varepsilon^3 \beta + 3\varepsilon^3 \langle u_1^2 V(\varepsilon, \hat{\mu}, \beta), v \rangle &= \mathcal{O}(\varepsilon^4).
\end{align*}

Implicit function theorem gives

\[\tilde{\mu} = \varepsilon^2 \hat{\mu} = \varepsilon^6 h(\varepsilon), \quad \beta = \varepsilon g(\varepsilon), \quad (H)\]

\(\varepsilon h(\varepsilon)\) and \(\varepsilon g(\varepsilon)\) are \(C^1\) functions of \(\varepsilon \in [0, \varepsilon_1]\).

Define "bad layers" of degree \(N\): \(BS_N(V) := \{(\varepsilon, \tilde{\mu}, \beta) \in \Lambda \times [-\beta_0, \beta_0]; \exists j; \tilde{\mu} \in (\tilde{\mu}_j^-(\varepsilon, \beta), \tilde{\mu}_j^+(\varepsilon, \beta))\}\).

In the 3-dimensional space \((\varepsilon, \tilde{\mu}, \beta)\) we need to check that the curve \((H)\) crosses transversally the bad set formed by the infinitely many thin layers \(\bigcup_{n \in \mathbb{N}} B_n S_{N_n}(V_{n-1})\), where \(N_n = [N_0(\gamma)]^{2^n}\), and \(V_n\) are the successive points obtained in the Newton iteration of the Nash-Moser method.
Transversality condition

The intersection of the surface $\beta = \varepsilon g(\varepsilon)$ with $\bigcup_{n \in \mathbb{N}} B_n S_N (V_{n-1})$ is a set of bad strips, Lipschitz continuous in $\varepsilon$.

**Transversality condition:** Let $\tilde{\mu}(\varepsilon)$ be any one of the limiting curves of the bad strips given by $\{\beta = \varepsilon g(\varepsilon)\} \cap \bigcap_{n \in \mathbb{N}} BS_N (V_{n-1})$.

Then we assume that for any of these curves, there exists $c > 0$ independent of $N$, such that for $h \in \mathbb{R}$ in a neighborhood of 0, the following inequality holds:

$$|\tilde{\mu}(\varepsilon + h) - \tilde{\mu}(\varepsilon)| \geq c|\varepsilon|^3|h|.$$  \hspace{1cm} (3)

NB: when eigenvalues $\sigma_j$ are not multiple, $\tilde{\mu}^{\pm}(\varepsilon, \beta(\varepsilon)$ has a slope $|t(\varepsilon)| > c_2|\varepsilon|$. So, the transversality condition is very weak. Moreover notice the slope of $(H)$ is of order $\varepsilon^5$. 
measure of "bad" \( \tilde{\mu}'s \) < \( C_\gamma |\varepsilon|^5 \) hence
measure of "bad" \( \varepsilon's \) < \( \frac{C_\gamma |\varepsilon|^5}{\min|t|} < \frac{C_\gamma |\varepsilon|^5}{c|\varepsilon|^3} \leq \frac{C_\gamma \varepsilon^2}{c} \).

The complementary subset in \((0, \varepsilon_2)\), is the good set of \( |\varepsilon| \), which is of asymptotic full measure since \([|\varepsilon| - \frac{C_\gamma \varepsilon^2}{c}] / |\varepsilon| \to 1 \) as \( \varepsilon \to 0 \).
Use a symmetry argument

\[ \tau U_\varepsilon = \beta_1 U_\varepsilon, \]

For \( \beta_1 = 1 \),

\[ \tau(\tilde{u}(\varepsilon) + \beta(\varepsilon)v) = \tau\tilde{u}(\varepsilon) + \beta(\varepsilon)w = \tilde{u}(\varepsilon) + \beta(\varepsilon)v \]

by the uniqueness of the solution \( u \). Hence, \( \beta(\varepsilon) \equiv 0 \).

For \( \beta_1 = -1 \),

\[ \tau\tilde{u}(\varepsilon) + \beta(\varepsilon)w = -\tilde{u}(\varepsilon) - \beta(\varepsilon)v \]

by the uniqueness of the solution \( -u \). This implies that in all cases

\[ \tau\tilde{u}(\varepsilon) = \beta_1\tilde{u}(\varepsilon), \quad \beta(\varepsilon) \equiv 0 \]
Existence of quasipatterns

**Theorem**

Assume \( \alpha \in \mathcal{E}_2 \cap \mathcal{E}_0 \cap \mathcal{E}_{qp} \) which is a full measure set, and assume that a transversality condition holds. Then there exist \( s_0 > 2 \) and \( \varepsilon_0 > 0 \) such that for an asymptotically full measure set of values of \( \varepsilon \in (0, \varepsilon_0) \), there exist two branches of bifurcating quasipattern solutions of SHE, invariant under rotation of angle \( \pi/3 \), of the form

\[
\begin{align*}
    u &= U_\varepsilon + \varepsilon^4 \tilde{u}(\varepsilon), \quad \tilde{u} \in \{v, w\}^\perp, \\
    U_\varepsilon &= \varepsilon(w + \beta_1 v) + \varepsilon^3 \tilde{u}_3, \quad \beta_1 = \pm 1, \tau u = \beta_1 u, \\
    \mu &= \varepsilon^2 \mu_2 + \varepsilon^4 \mu_4 + \tilde{\mu}(\varepsilon),
\end{align*}
\]

where \( \tilde{u}(\varepsilon) \in Q_0 \mathcal{H}_{s_0} \), \( w, v, \tilde{u}_3, \mu_2, \mu_4 \) are defined above, and functions of \( \varepsilon \) are \( C^1 \) with \( \tilde{u}(0) = 0 \), \( \tilde{\mu}(\varepsilon) = O(\varepsilon^6) \). \( Su = -u \) corresponds to change \( \varepsilon \) into \( -\varepsilon \).


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