Linking with Jean-Marc

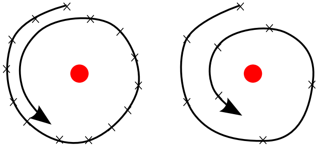
Nice, June 2018

Étienne Ghys

Cruis (Alpes de Haute Provence), October 1992

Theorem : (Naishul, 1982) : *Let F_1, F_2 be **two area preserving diffeomorphisms** defined in a neighborhood of the origin of* R^2*. Suppose that their differentials at the origin are rotations of angles* α_1, α_2*. Suppose that they are conjugate by some (orientation preserving) **homeomorphism**. Then* $\alpha_1 = \alpha_2$*.*

Theorem : (Naishul, 1982) : *Let F_1, F_2 be* ***two area preserving diffeomorphisms*** *defined in a neighborhood of the origin of $\mathbb{R}^2$. Suppose that their differentials at the origin are rotations of angles α_1, α_2. Suppose that they are conjugate by some (orientation preserving)* ***homeomorphism****. Then $\alpha_1 = \alpha_2$.*



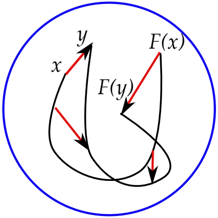
$(r;\theta) \mapsto (r - r^2, \theta + \alpha_1)$ and $(r;\theta) \mapsto (r - r^2, \theta + \alpha_2)$ are topologically conjugate.

Calabi and Ruelle

Let F be an area preserving diffeomorphism of the disc D which is the identity in the neighborhood of the boundary. Choose some isotopy F_t from $F_0 = id$ to $F_1 = F$.

If $x, y \in D$, denote by $Ang_F(x, y) \in \mathbf{R}$ the variation of the argument of $F_t(y) - F_t(x)$ when t goes from 0 to 1.

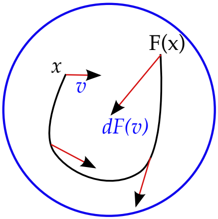
Definition : The Calabi invariant of F is $\iint Ang_F(x, y)\, dx\, dy$.



Let F be an area preserving diffeomorphism of the disc D which is the identity in the neighborhood of the boundary. Choose some isotopy F_t from $F_0 = id$ to $F_1 = F$.

If $x \in D$, denote by $A_F(x, v)$ the variation of the argument of $dF_t(x)(v)$ when t goes from 0 to 1 for some vector v. Denote by $A_F(x)$ the limit $\frac{1}{n} A_{F^n}(x, v)$.

Definition : The Ruelle invariant of F is $\int A_F(x)\, dx$.



Theorem : *Calabi and Ruelle are both invariant by area preserving homeomorphisms.*

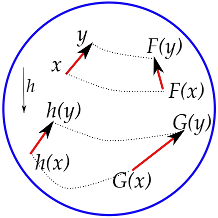
- Calabi and Ruelle are not defined for area preserving homeomorphims.
- Calabi is the **homomorphism** $Diff_0(D, \partial D, area) \to \mathbf{R}$.
- Is $Homeo_0(D, \partial D, area)$ simple ?

- Calabi and Ruelle are not defined for area preserving homeomorphims.
- Calabi is the **homomorphism** $Diff_0(D, \partial D, area) \to \mathbf{R}$.
- Is $Homeo_0(D, \partial D, area)$ simple ?

- Calabi and Ruelle are not defined for area preserving homeomorphims.
- Calabi is the **homomorphism** $Diff_0(D, \partial D, area) \to \mathbf{R}$.
- Is $Homeo_0(D, \partial D, area)$ simple ?

Let $G = hFh^{-1}$.

$$Ang_F(x, y) = Ang_h(x, y) + Ang_G(h(x), h(y)) - Ang_h(G(x), G(y))$$

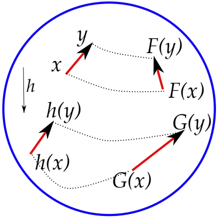


If Ang_h would be integrable, $\iint Ang_F = \iint Ang_G$.

$(r, \theta) \mapsto (r, \theta + \exp(1/r))$ is a non smooth homeo !

Let $G = hFh^{-1}$.

$$Ang_F(x,y) = Ang_h(x,y) + Ang_G(h(x),h(y)) - Ang_h(G(x),G(y))$$

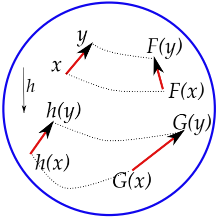


If Ang_h would be integrable, $\iint Ang_F = \iint Ang_G$.

$(r,\theta) \mapsto (r, \theta + \exp(1/r))$ is a non smooth homeo !

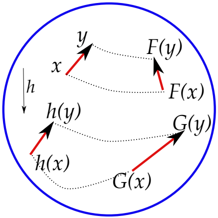
Let $G = hFh^{-1}$.

$$Ang_F(x, y) = Ang_h(x, y) + Ang_G(h(x), h(y)) - Ang_h(G(x), G(y))$$



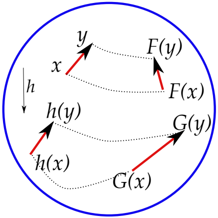
If Ang_h would be integrable, $\iint Ang_F = \iint Ang_G$.

$(r, \theta) \mapsto (r, \theta + \exp(1/r))$ is a non smooth homeo !



Poincaré recurrence.

$$
\begin{aligned}
Cal(F) - Cal(G) &= \left(Cal(F^n) - Cal(G^n)\right)/n \\
&= \iint \left(Ang_{F^n}(x,y) - Ang_{G^n}(h(x),h(y))\right)/n \\
&= \iint \left(Ang_h(x,y) - Ang_h(G^n(x),G^n(y))\right)/n \\
&= 0
\end{aligned}
$$

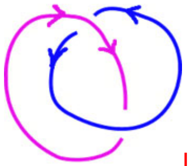


Poincaré recurrence.

$$\begin{aligned}
Cal(F) - Cal(G) &= \left(Cal(F^n) - Cal(G^n)\right)/n \\
&= \iint \left(Ang_{F^n}(x,y) - Ang_{G^n}(h(x),h(y))\right)/n \\
&= \iint \left(Ang_h(x,y) - Ang_h(G^n(x),G^n(y))\right)/n \\
&= 0
\end{aligned}$$

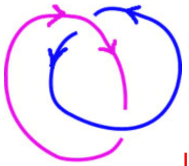
Linking and braiding

Linking numbers



Linking $= +1$

Linking numbers

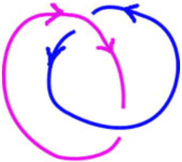


Linking = +1

-1

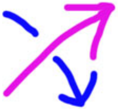
$+1.$

Linking numbers

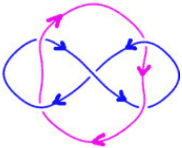


Linking = +1

-1

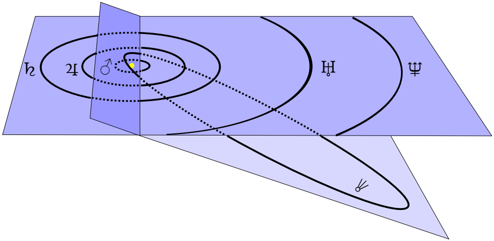


$+1.$



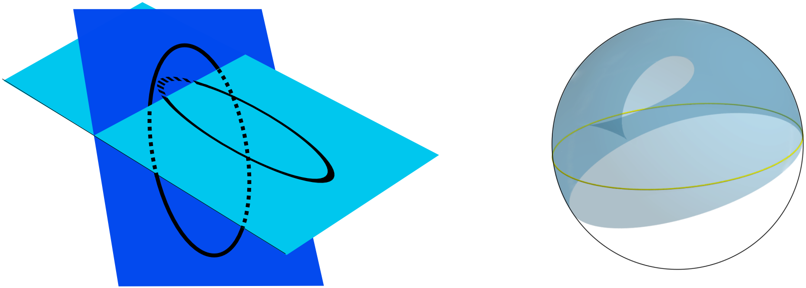
Linking = 0

Gauss Zodiacus



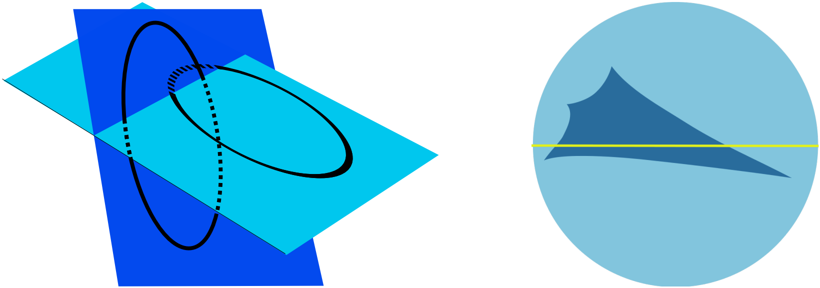
The Zodiacus map $\mathbf{S}^1 \times \mathbf{S}^1 \to \mathbf{S}^2$.

Gauss Zodiacus



Unlinked ellipses and the associated zodiacus

Gauss Zodiacus



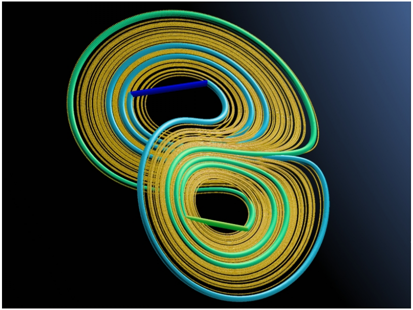
Linked ellipses and the associated zodiacus *linking* = integral of the Jacobian of the zodiacus

$$linking(x(t), y(s)) = \int\int \frac{\det(\frac{dx}{dt}, \frac{dy}{ds}, x(t) - y(s))}{\|x(t) - y(s)\|^3}\, dtds$$

linking = integral of the Jacobian of the zodiacus

$$linking(x(t), y(s)) = \int\int \frac{\det(\frac{dx}{dt}, \frac{dy}{ds}, x(t) - y(s))}{\|x(t) - y(s)\|^3}) \, dtds$$

Arnold's invariant



X : a volume preserving vector field on the 3-sphere, generating a dynamical system ϕ_t.

$$k(x, T) = [x \xrightarrow{\phi} \phi_T(x) \xrightarrow{\textit{segment}} x]$$

$$\textit{linking}\,(k(x_1, T_1), k(x_2, T_2))$$

Arnold's invariant for a flow in the 3-sphere.

Theorem : *Let X be a volume preserving vector field on the 3-sphere, generating a dynamical system ϕ_t.* Then

$$\mathit{Helicity}(x_1, x_2) := \lim_{T_1, T_2 \to \infty} \frac{1}{T_1 T_2} \mathit{linking}\left(k(x_1, T_1), k(x_2, T_2)\right).$$

exists for Lebesgue almost every (x_1, x_2). The integral

$\iint \mathit{Helicity}(x_1, x_2) dx_1 \, dx_2$ is the Helicity of X.

(or Hopf, or Moreau, or Moffatt, or Arnold invariant invariant of X).

Arnold's invariant for a flow in the 3-sphere.

Theorem : *Let X be a volume preserving vector field on the 3-sphere.*

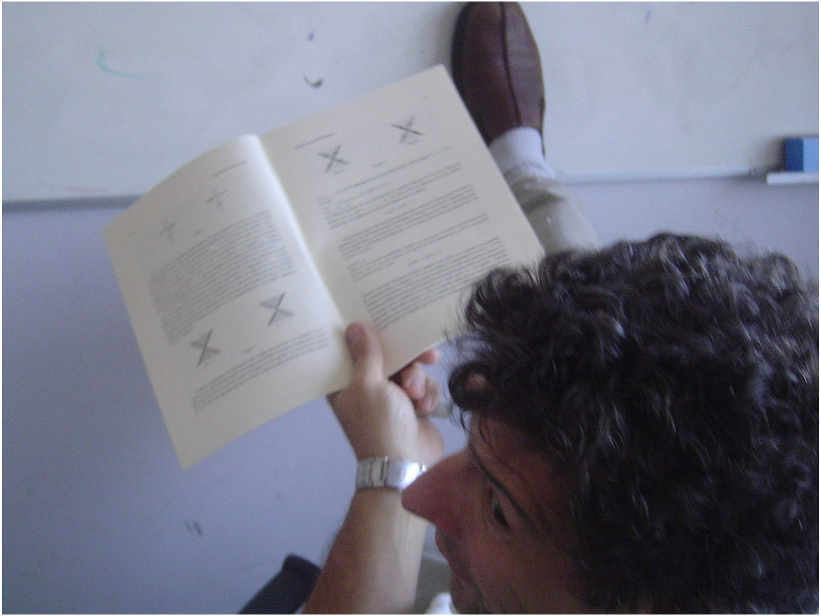
$i_X vol = d\alpha$

The helicity of X is $\int \alpha \wedge d\alpha$.

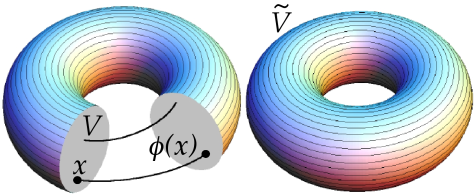
Arnold's invariant for a flow in the 3-sphere.

Conjecture (Arnold)

Two volume preserving vector fields which are conjugate by a homeomorphism which preserves the volume (and the orientation) have the same helicity.

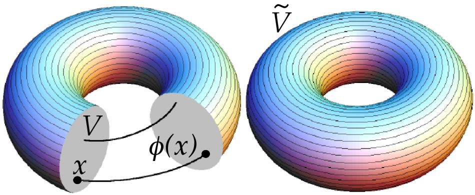


Theorem : *Calabi of a diffeo = Arnold of its suspension.*



Corollary : *Arnold's conjecture is true for suspensions.*

Theorem : *Calabi of a diffeo = Arnold of its suspension.*

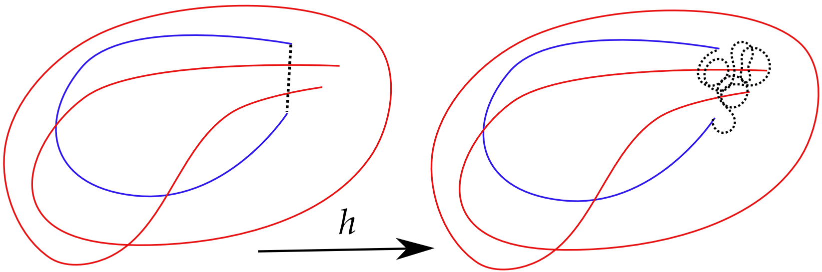


Corollary : *Arnold's conjecture is true for suspensions.*

Why this conjecture is difficult...

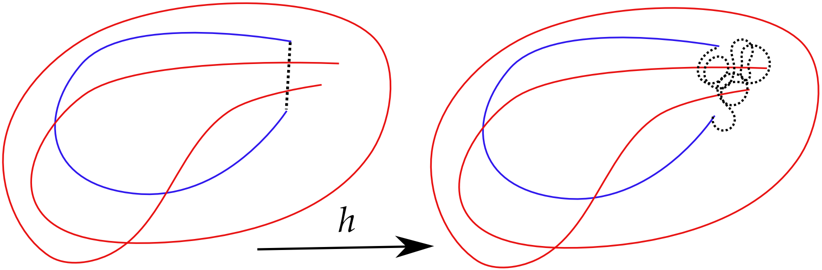
The linking number of two disjoint closed curves is a topological invariant...

but a homeomorphism might send a segment to a very crooked fractal curve which might link in some non controlled way with the orbits.



Why this conjecture is difficult...

The linking number of two disjoint closed curves is a topological invariant...
but a homeomorphism might send a segment to a very crooked fractal curve which might link in some non controlled way with the orbits.

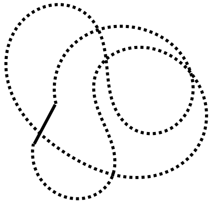


Why this conjecture is difficult...

Comparing the helicity of two topologically equivalent flows ϕ_T and ψ_T, one has to compare two cuvres :

- the image by h of the segment connecting x_1 and $\phi_{T_1}(x_1)$.
- the segment between $h(x_1)$ and $h(\phi_{T_1}(x_1)) = \psi_{T_1}(h(x_1)$

This defines a fractal closed curve



Then, one has to get an upper bound of the linking number of this fractal curve with long pieces of orbits of ψ.

Left handed vector fields

Definition : *A vector field X on the 3-sphere, generating a dynamical system ϕ_t, is called left handed if for any two distinct points x_1, x_2 the trajectories starting from x_1 and x_2 link positively :*

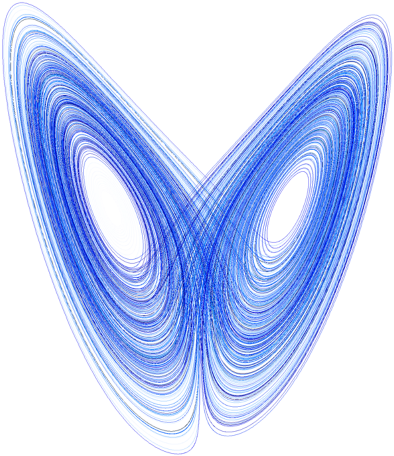
$$\liminf_{T_1, T_2 \to \infty} \frac{1}{T_1 T_2} \mathit{linking}\,(k(x_1, T_1), k(x_2, T_2)) > 0.$$

Stable property : Small C^1 perturbations of left handed vector fields are left handed.

Definition : *A vector field X on the 3-sphere, generating a dynamical system ϕ_t, is called left handed if for any two distinct points x_1, x_2 the trajectories starting from x_1 and x_2 link positively :*

$$\liminf_{T_1, T_2 \to \infty} \frac{1}{T_1 T_2} \textit{linking}\,(k(x_1, T_1), k(x_2, T_2)) > 0.$$

Stable property : Small C^1 perturbations of left handed vector fields are left handed.



The Lorenz attractor is left handed.

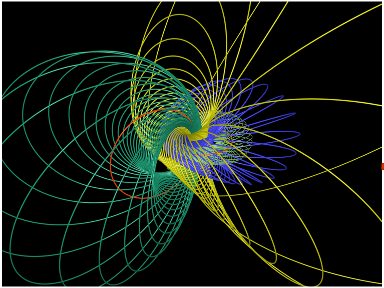
Example

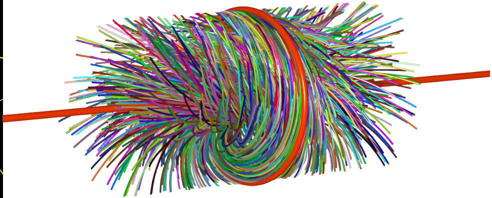
Two independent oscillators : dynamics on $\mathbf{C}^2 \simeq \mathbf{R}^4$:

$$\phi^t(z_1, z_2) = (\exp(it)z_1, \exp(it)z_2)$$

- $|z_1|^2 + |z_2|^2$ is invariant, so the dynamics is on spheres $\mathbf{S}^3$ in $\mathbf{R}^4$.
- Orbits are the fibers of the Hopf fibration :

$$(z_1, z_2) \in \mathbf{S}^3 \subset \mathbf{C}^2 \mapsto \frac{z_1}{z_2} \in \mathbf{C} \cup \{\infty\} \simeq \mathbf{S}^2.$$





Characterization

Let $\mathcal{P}$ be the compact convex set of probability measures which are invariant under the flow (for instance, periodic orbits).

There is a well defined quadratic linking form :

$$\mathit{link} : (\mu_1, \mu_2) \in \mathcal{P} \times \mathcal{P} \mapsto \mathit{linking}(\mu_1, \mu_2) \in \mathbf{R}$$

Theorem : *A vector field X on the 3-sphere is left handed if and only if the linking quadratic form is positive.*

Characterization

Let $\mathcal{P}$ be the compact convex set of probability measures which are invariant under the flow (for instance, periodic orbits).

There is a well defined quadratic linking form :

$$\textit{link} : (\mu_1, \mu_2) \in \mathcal{P} \times \mathcal{P} \mapsto \textit{linking}(\mu_1, \mu_2) \in \mathbf{R}$$

Theorem : *A vector field X on the 3-sphere is left handed if and only if the linking quadratic form is positive.*

Characterization

Let $\mathcal{P}$ be the compact convex set of probability measures which are invariant under the flow (for instance, periodic orbits).

There is a well defined quadratic linking form :

$$\mathit{link} : (\mu_1, \mu_2) \in \mathcal{P} \times \mathcal{P} \mapsto \mathit{linking}(\mu_1, \mu_2) \in \mathbf{R}$$

Theorem : *A vector field X on the 3-sphere is left handed if and only if the linking quadratic form is positive.*

Theorem : *A vector field X on the 3-sphere is left handed if and only there is a Gauss linking form Ω which is positive on X.*

$$linking(x(t), y(s)) = \int\int \frac{\det\left(\frac{dx}{dt}, \frac{dy}{ds}, x(t) - y(s)\right)}{\|x(t) - y(s)\|^3} dtds$$

$$linking(x(t), y(s)) = \int\int \Omega_{x(t),y(s)}\left(\frac{dx}{dt}, \frac{dy}{ds}\right) dtds$$

Theorem : *A vector field X on the 3-sphere is left handed if and only there is a Gauss linking form Ω which is positive on X.*

$$linking(x(t), y(s)) = \int\int \frac{\det\left(\frac{dx}{dt}, \frac{dy}{ds}, x(t) - y(s)\right)}{\|x(t) - y(s)\|^3} dtds$$

$$linking(x(t), y(s)) = \int\int \Omega_{x(t),y(s)}\left(\frac{dx}{dt}, \frac{dy}{ds}\right) dtds$$

Theorem : *A vector field X on the 3-sphere is left handed if and only there is a Gauss linking form Ω which is positive on X.*

$$linking(x(t), y(s)) = \int\int \frac{\det\left(\frac{dx}{dt}, \frac{dy}{ds}, x(t) - y(s)\right)}{\|x(t) - y(s)\|^3} \, dtds$$

$$linking(x(t), y(s)) = \int\int \Omega_{x(t),y(s)}\left(\frac{dx}{dt}, \frac{dy}{ds}\right) dtds$$

Any finite collection of periodic orbits is the binding of some Birkhoff section, an open book transverse to the vector field.

Marie Lhuissier : Suppose a flow has a Birkhoff section which is a disc with a first return map F. Then the flow is left handed if and only if F is "turning" : a weak version of positivity for $Ang_F(x, y)$.

Three conjectures

Three conjectures

- The geodesic flow of a convex surface S, on $\tilde{T}^1(S)$ is left handed.
- The dynamics of a point in the plane under a convex potential $U(x, y)$ is left handed.
- The Restricted Circular Planar 3 body Problem is left handed up to the first Lagrange point.

Three conjectures

- The geodesic flow of a convex surface S, on $\tilde{T}^1(S)$ is left handed.
- The dynamics of a point in the plane under a convex potential $U(x, y)$ is left handed.
- The Restricted Circular Planar 3 body Problem is left handed up to the first Lagrange point.

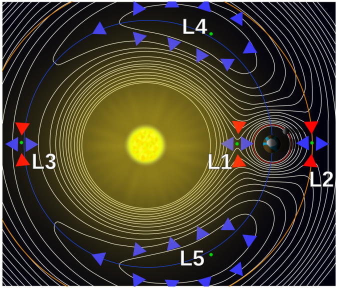
Three conjectures

- The geodesic flow of a convex surface S, on $\tilde{T}^1(S)$ is left handed.
- The dynamics of a point in the plane under a convex potential $U(x, y)$ is left handed.
- The Restricted Circular Planar 3 body Problem is left handed up to the first Lagrange point.

3 body in a rotating frame

$$\frac{1}{2}(u^2 + v^2) - V(x, y) = Constant = h$$

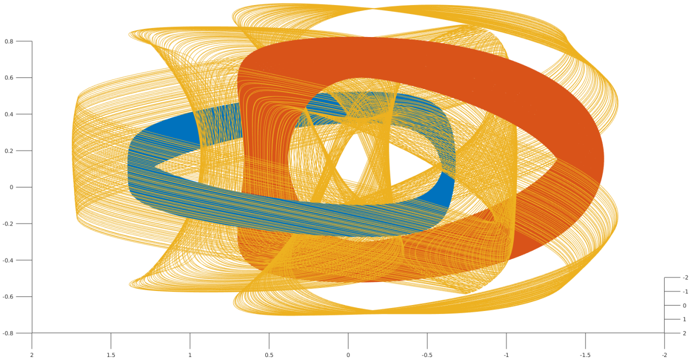
$$V(x, y) = \frac{1-\mu}{\sqrt{(x+\mu)^2 + y^2}} + \frac{\mu}{\sqrt{(x+\mu-1)^2 + y^2}} + \frac{1}{2}(x^2 + y^2)$$



© Wikipedia

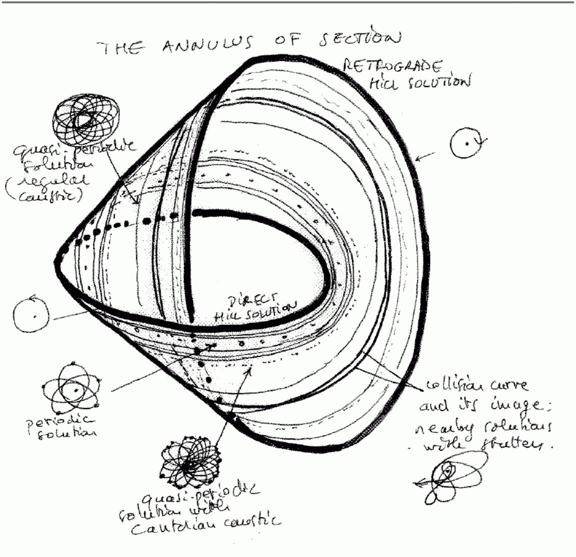
3 body in a rotating frame

If the energy is $<$ than the energy of the first Lagrange point, the configuration space, after desingularization, is a 3-sphere.



© Lhuissier

If the energy is << 0 there is a Birkhoff section (Poincaré)



The PR3BP seems to be left handed, at least up to the first Lagrange point (Lhuissier).

```
load('atable.mat','T1')
T1
% mu : parametre de masse   ( corps de masses 1-mu,mu en -mu,-mu+1)
% delta : ecart a l'energie du point de Lagrange L1
% enl(min,max,moy) : enlacement(min,max,moy) normalise par unite de temps^2
% calcule sur nbenl couples de trajectoires
% tenl : taux d'enlacement sur le niveau d'energie (invariant d'Arnold
% normalise par le volume au carr) calcule par la formule integrale
```

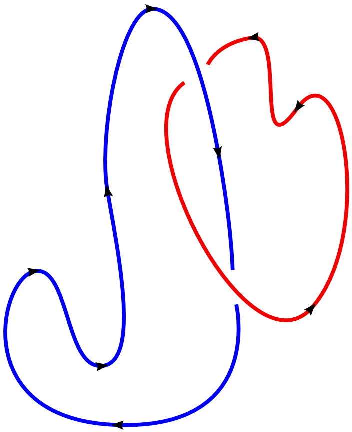
T1 =

708 table

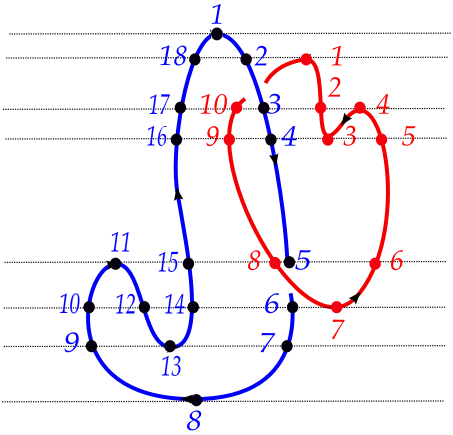
mu	delta	E0	enlmin	enlmax	enlmoy	tenl	nbenl
0.1	0.001	-1.7995	0.14122	1.0831	0.40445	0.38961	866
0.1	0.01	-1.8085	0.14658	1.0911	0.34332	0.40153	892
0.1	0.1	-1.8985	0.20524	1.217	0.51279	0.51388	948
0.1	0.15	-1.9485	0.24948	1.3026	0.53924	0.57872	625
0.1	0.2	-1.9985	0.29186	1.4666	0.62804	0.64643	957
0.1	0.25	-2.0485	0.36077	1.5293	0.7076	0.71726	541
0.1	0.3	-2.0985	0.40807	1.7233	0.87044	0.79141	588
0.1	0.35	-2.1485	0.47224	1.7325	0.89485	0.86914	568
0.1	0.4	-2.1985	0.49441	1.8661	0.954	0.9505	549
0.1	0.45	-2.2485	0.56716	1.9891	1.0836	1.0356	568
0.1	0.5	-2.2985	0.050644	2.1112	0.96931	1.1247	976
0.1	1	-2.7985	1.489	3.5378	2.3216	2.2535	533
0.2	0.001	-1.9033	0.22144	1.2569	0.46926	0.52154	685
0.2	0.01	-1.9123	0.24165	1.3658	0.54993	0.53676	704
0.2	0.05	-1.9523	0.26932	0.70626	0.423	0.59925	171
0.2	0.1	-2.0023	0.31328	1.392	0.63835	0.67747	857
0.2	0.15	-2.0523	0.36021	1.5941	0.71101	0.75836	786
0.2	0.2	-2.1023	0.41736	1.6976	0.81927	0.84276	667
0.2	0.25	-2.1523	0.51341	1.7925	0.90636	0.93118	694
0.2	0.3	-2.2023	0.55281	1.9485	0.9826	1.0238	515
0.2	0.35	-2.2523	0.63514	2.1411	1.1952	1.1208	351
0.2	0.4	-2.3023	0.72592	2.154	1.2596	1.2225	406
0.2	0.45	-2.3523	0.89235	2.3342	1.4531	1.3291	378
0.2	0.5	-2.4023	0.85778	2.5035	1.3684	1.4405	722
0.2	1	-2.9023	1.9829	4.0518	2.8353	2.8553	532
0.3	0.001	-1.9611	0.31059	1.5285	0.68452	0.62269	463
0.3	0.01	-1.9701	0.29872	1.4582	0.6008	0.64139	478
0.3	0.1	-2.0601	0.4067	1.5997	0.82247	0.81195	459
0.3	0.15	-2.1101	0.47069	1.7421	0.89633	0.90987	435

Back to Arnold's conjecture

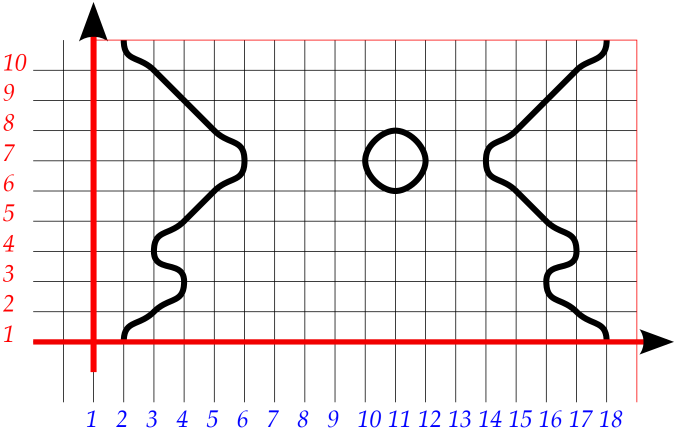
Another way of computing the linking number.



Another way of computing the linking number.



Another way of computing the linking number.



The set of pairs of points in $k_1 \times k_2$ with the same z-coordinate is a 1-dimensional curve $C(k_1, k_2)$ in a 2-dimensional torus. The linking number is the degree of the obvious map $C(k_1, k_2) \to \mathbf{S}^1$.

If ϕ_t is a volume preserving flow on S^3, consider a generic linear projection $\pi : \mathbf{S}^3 \to \mathbf{R}$ and consider the set C of pairs of points with the same π projection.

This is a 5-dimensional object (the topological suspension of $\mathbf{S}^2 \times \mathbf{S}^2$).

There is a volume preserving flow Φ_t on C and a map $C \to \mathbf{S}^1$.

The Arnold invariant is the asymptotic rotation number of Φ_t.

If ϕ_t is a volume preserving flow on S^3, consider a generic linear projection $\pi : \mathbf{S}^3 \to \mathbf{R}$ and consider the set C of pairs of points with the same π projection.

This is a 5-dimensional object (the topological suspension of $\mathbf{S}^2 \times \mathbf{S}^2$).

There is a volume preserving flow Φ_t on C and a map $C \to \mathbf{S}^1$.

The Arnold invariant is the asymptotic rotation number of Φ_t.

If ϕ_t is a volume preserving flow on S^3, consider a generic linear projection $\pi : \mathbf{S}^3 \to \mathbf{R}$ and consider the set C of pairs of points with the same π projection.

This is a 5-dimensional object (the topological suspension of $\mathbf{S}^2 \times \mathbf{S}^2$).

There is a volume preserving flow Φ_t on C and a map $C \to \mathbf{S}^1$.

The Arnold invariant is the asymptotic rotation number of Φ_t.

If ϕ_t is a volume preserving flow on S^3, consider a generic linear projection $\pi : \mathbf{S}^3 \to \mathbf{R}$ and consider the set C of pairs of points with the same π projection.

This is a 5-dimensional object (the topological suspension of $\mathbf{S}^2 \times \mathbf{S}^2$).

There is a volume preserving flow Φ_t on C and a map $C \to \mathbf{S}^1$.

The Arnold invariant is the asymptotic rotation number of Φ_t.

Bon anniversaire Jean-Marc !