Cellular automata, tilings and undecidability

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June 14, 2018

Cellular automata

$\langle S, D, r, f \rangle$

$S =$ set of states

$D =$ dimension

$r =$ radius

$f \colon S^{(2r+1)^D} \to S$

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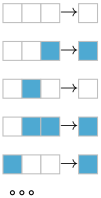

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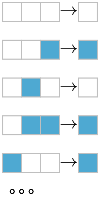

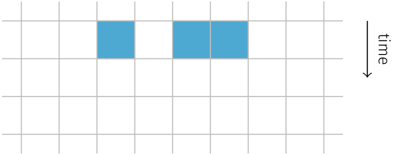

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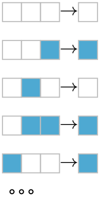

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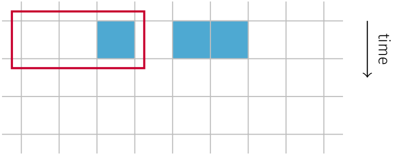

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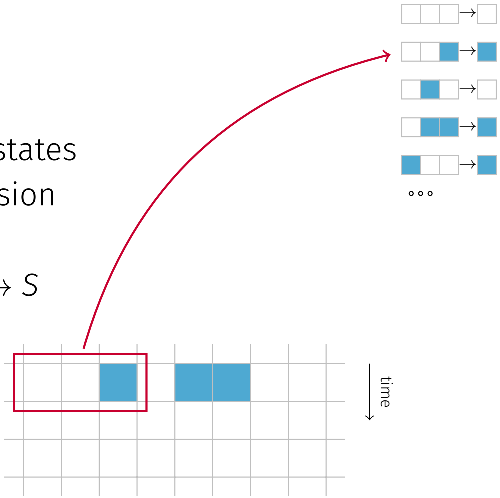

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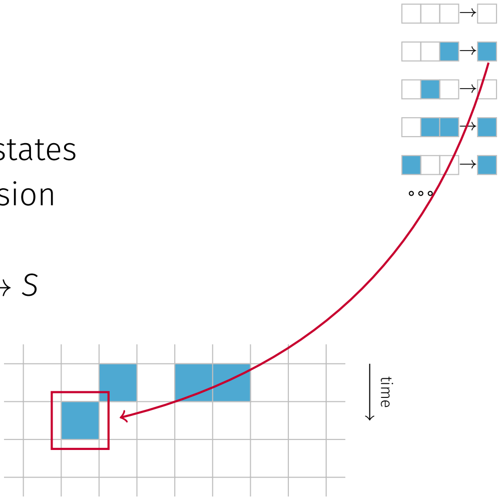

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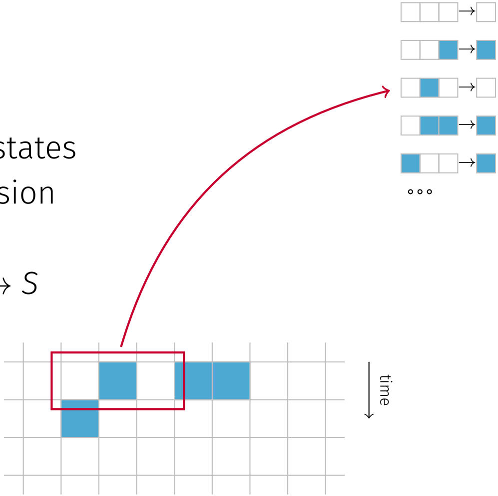

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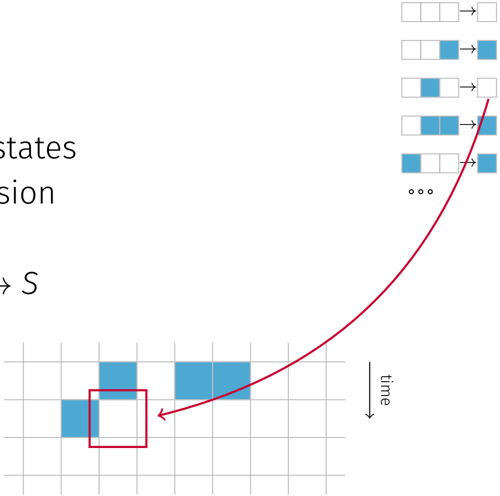

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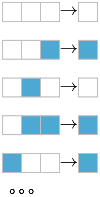

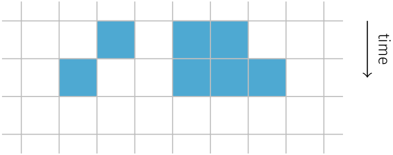

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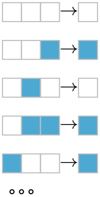

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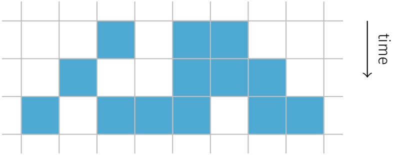

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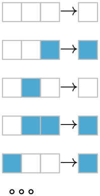

$\langle S, D, r, f \rangle$

$S =$ set of states

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$f\colon S^{(2r+1)^D} \to S$

$$\forall c \in S^{\mathbb{Z}^D} \forall \vec{x} \in \mathbb{Z}^D \; F(c)_{\vec{x}} = f(M_r^{\vec{x}})$$

Fact

CA are symbolic dynamical systems.

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Definition (Closingness)

A 1D CA F is *closing* if it is injective over left or right asymptotic asymptotic pairs.

Definition (Asymptotic pairs)

Two configurations x, y are *left-asymptotic* (resp. *right-asymptotic*) if there exists some $k \in \mathbb{Z}$ such that for all $n > k$ (resp. $n < k$) it holds $x_n = y_n$.

Fact

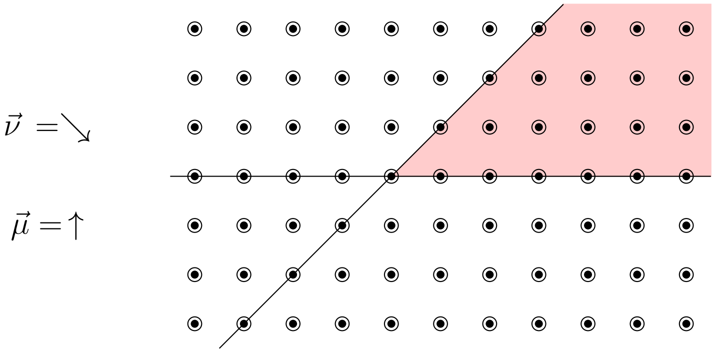
This definition of closingness is pointless in $D \geq 2$.

Closingness for $D \geq 2$

Definition

Given two (non-collinear vectors) $\vec{\mu}, \vec{\nu} \in \mathbb{Z}^D$, two configurations c, c' are μ(resp. $\vec{\mu}$-$\vec{\nu}^-$, resp. $\vec{\mu}$-$\vec{\nu}^+$)-*asymptotic* iff there exist $q \in \mathbb{Z}$ (resp. $q, q' \in \mathbb{Z}$) such that for all $\vec{x} \in \mathbb{Z}^d$ such that $\vec{\mu} \cdot \vec{x} \geq q$ (resp. $\vec{\mu} \cdot \vec{x} \geq q$ **and** $\vec{\nu} \cdot \vec{x} \geq q'$, resp. $\vec{\mu} \cdot \vec{x} \geq q$ **or** $\vec{\nu} \cdot \vec{x} \geq q'$) it holds $c(\vec{x}) = c'(\vec{x})$.

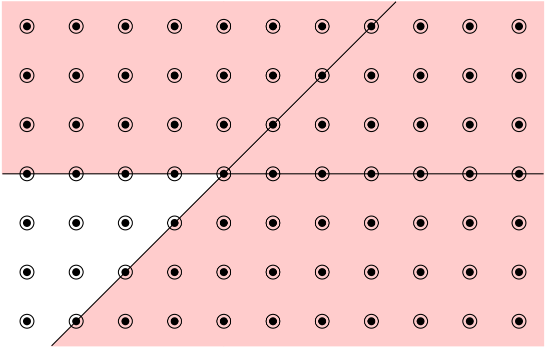
Closingness for $D \geq 2$



Closingness for $D \geq 2$

$\vec{\nu} = \searrow$

$\vec{\mu} = \uparrow$

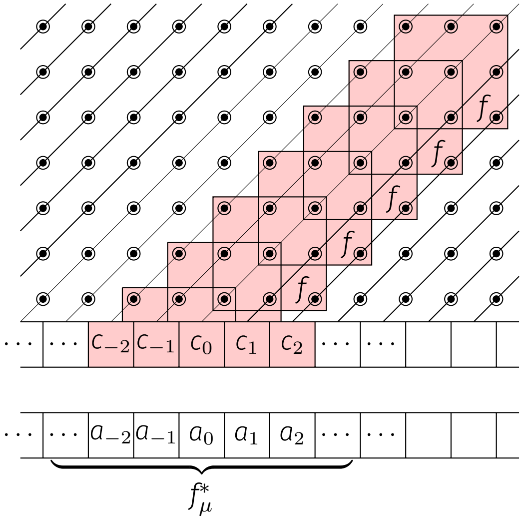


Some results

Theorem (A)
For $D \geq 2$, bi-closing D-dimensional CA are open, surjective and have JDPO property.

Theorem (B)
For $D \geq 2$, closingness is undecidable.

The slicing construction



Slicing construction (sequel)

Lemma (Slicing lemma)

For any $\mu \in \mathbb{Z}^D$ ($D \geq 2$), F and F^*_μ are topologically semi-conjugated.

Lemma (Slicing invariance)

For any $\mu, \nu \in \mathbb{Z}^D$ ($D \geq 2$) and any D-dimensional CA F, F^*_μ and F^*_ν are topologically conjugated.

Tilings

Fix a finite set of colors
$c = \{\blacksquare, \blacksquare, \dots, \blacksquare\}$

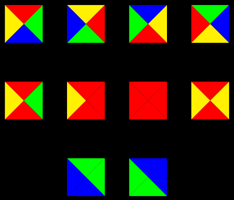
Definition (Wang tiles)

A *Wang tile* t is a map from $\{N, E, S, W\}$ to c.

Definition (Wang tiles set)

It is a set of Wang tiles defined on the same (finite) color set.

An example



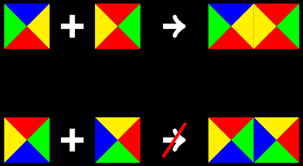
Tilings

Fix a finite set of Wang tiles T.

Definition (Tiling)

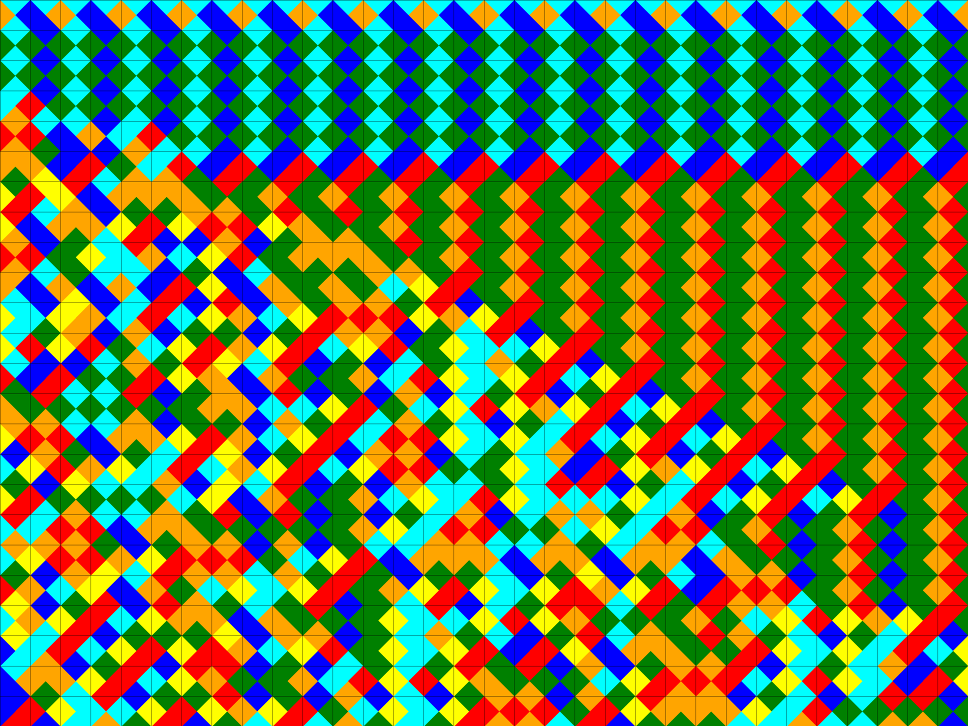
A *tiling* τ (based on T) is a map from T to $\mathbb{Z}^2$ satisfying the local matching property.

Local matching property



No rotations

No flipping



Problem (Domino tiling problem)

Given a (finite) set of Wang tiles τ, does τ admit a tiling?

Theorem (Berger 1966)

The domino tiling problem is undecidable.

Ideas for the proof.

- There is an algorithms that recursively enumerates tile sets that do not tile the whole plane;
- There is an algorithm that recursively enumerates tile sets that tile the plane periodically.
- There exists a tile set that tiles the plane but never periodically.

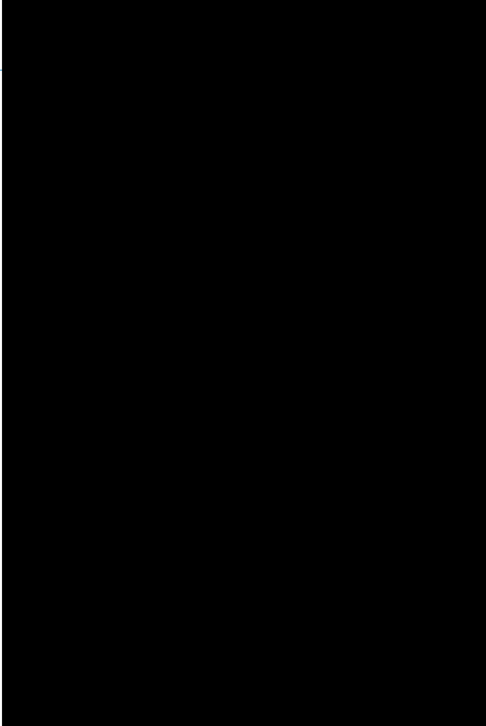
□

The quest for aperiodic tile sets

26426	Berger, 1966
104	Berger, shortly after
92	Knuth, 1966
56	Robinson, 1967
35	Robinson, 1971
34	Penrose, 1973
32	Robinson, 1973
24	Robinson, 1977
16	Amman, 1978
13	Culik and Kari, 1995
11	Jeandel and Rao, 2015

Kari's construction

- a tile set τ
- Cross tiling
- Peano tiling
- $\{0, 1\}$ labels



Kari's construction

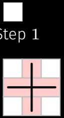
- a tile set τ
- Cross tiling
- Peano tiling
- $\{0, 1\}$ labels

Idea: exploit hidden details to reduce closingness to the domino tiling problem

Kari’s construction

Step 1

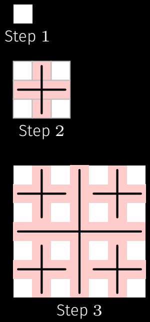
Kari’s construction



Step 1

Step 2

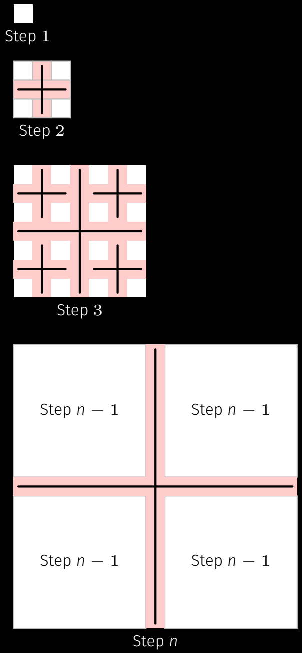
Kari’s construction



Kari's construction

Lemma (Kari, 1994)

The cross tiling is aperiodic.



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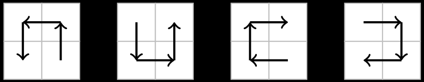
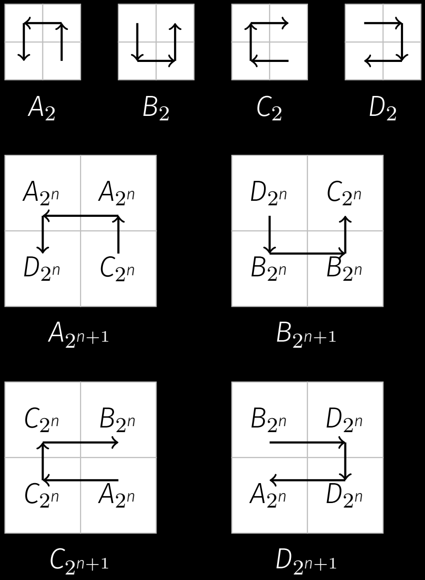


A_2 B_2 C_2 D_2

Kari's construction



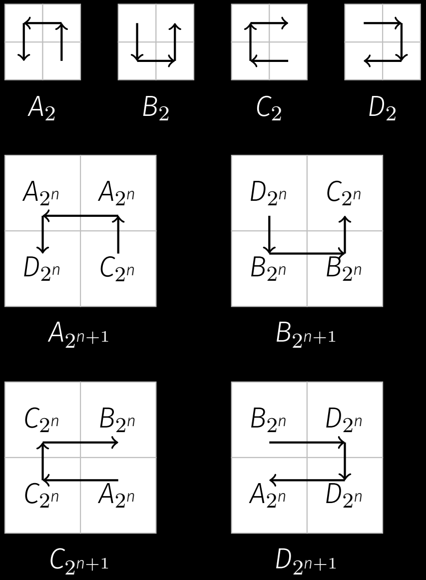
Kari's construction

Lemma (Kari, 1994)

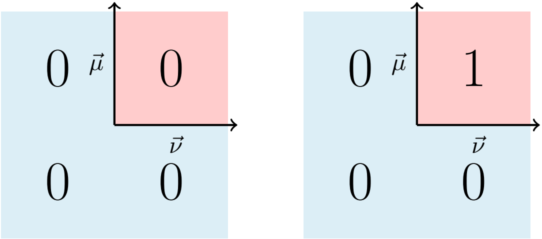
The Peano tiling produces a space filling path.

Hidden details:

- 1 space filling path;
- 2 space filling paths;
- 4 space filling paths.

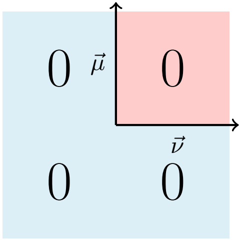


Back to closingness



configuration c_0 configuration c_1

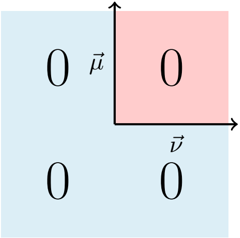
Back to closingness



configuration c_0 configuration c_1

- update labels using 'nearest xor' along the paths

Back to closingness



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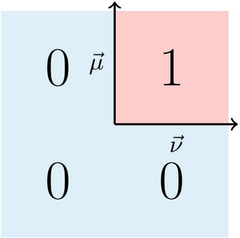
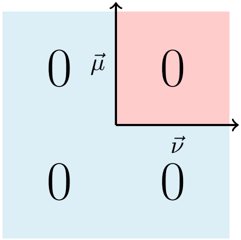


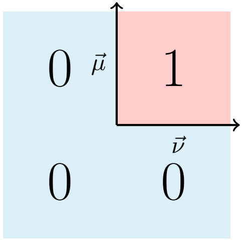
configuration c_1

- update labels using 'nearest xor' along the paths
- do nothing if there is a tiling error

Back to closingness



configuration c_0



configuration c_1

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Conclusion: $\vec{\mu}$-$\vec{\nu}$-closing iff there is a tiling error (on τ).

Questions

- Undecidability of (positive) expansivity?
- What kind of tilings can CA produce?
- Can CA produce really complex tilings?
- …

Thank you.