# Strongly dissipative diffeomorphisms of the disc with zero entropy

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Joint work with Enrique Pujals and Charles Tresser.

*Systèmes dynamiques et systèmes complexes*

Nice, june 12-14th 2018

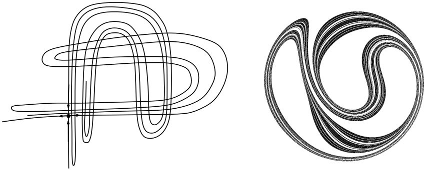
# Dynamics of surface diffeomorphisms

$f$: a $C^2$-diffeomorphism of the 2-disc $\mathbb{D}$.

$h_{top}(f)$: its topological entropy.

(exponential growth rate of the number of orbits)

When $h_{top}(f) > 0$,
the dynamics is rich.



Jean-Marc has studied the following questions:

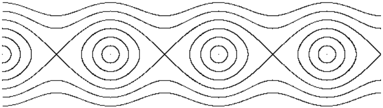
*How is the dynamics when $h_{top}(f) = 0$?*

*What happens just before the entropy becomes positive?*

# Examples with zero entropy

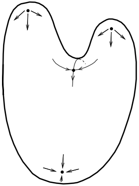
Conservative examples:

- translations on $\mathbb{T}^2$,
- hamiltonian systems.

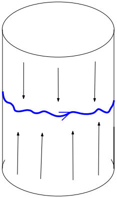


Dissipative examples:

Morse-Smale systems



irrational attractors



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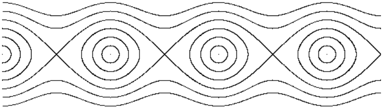

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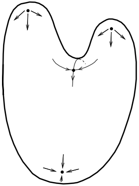

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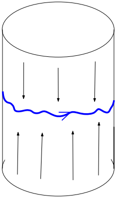
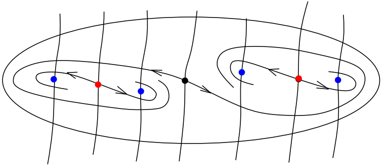


Period doubling cascade and odometer.

(Gambaudo - Tresser - Van Strien)



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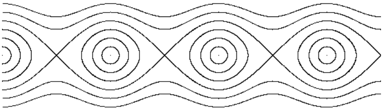

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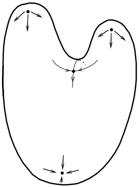

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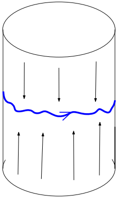

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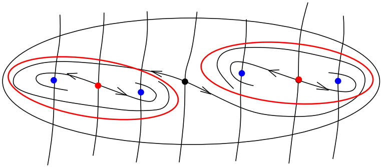

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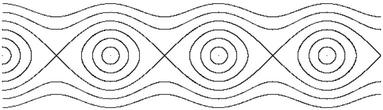

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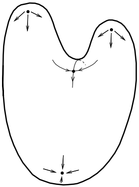

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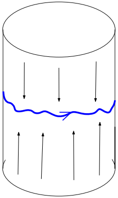

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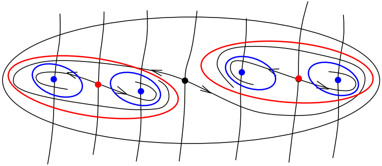

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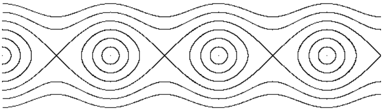

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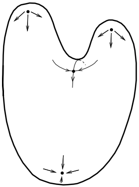

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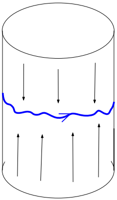

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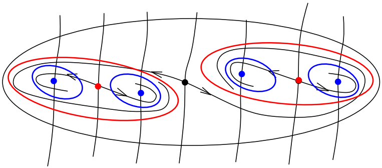

*Some converse results:* Franks-Handel (conservative), Le Calvez-Tal. See next talk!

# Set of periods when the entropy vanishes

$n$ is a *period* of $f$ if $f^n$ has a fixed point $x$, not fixed by $f^m$, $m < n$.

**Dimension** 1. Endomorphisms of $[0, 1]$.

| |
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| Sharkovskii's thm. *The set of periods (if infinite) is* $\{2^n, n \in \mathbb{N}\}$. |

Coullet-Tresser / Feigenbaum. *The dynamics is renormalizable.*

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**Dimension 2.** Diffeomorphisms of the disc.

No direct generalization of Sharkovskii!

Gambaudo and co-authors. *Forcing braids types of periodic orbits.*

Gambaudo-Tresser conjecture. *In the dissipative case,*
– *there exists* $k_0 \geq 1$ *such that any period has the form* $k.2^n$, $k \leq k_0$.
– *the set of periods (if infinite) contains* $\{k.2^n, n \in \mathbb{N}\}$ *for some* $k$.

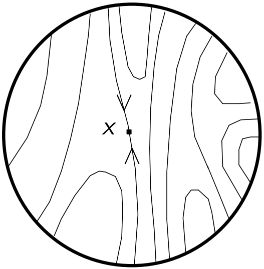
De Carvalho-Martens-Lyubich. *Renormalization for Hénon maps with jacobian* $\ll 1$.

# Strongly dissipative diffeomorphisms

A diffeomorphism $f$ is *strongly dissipative* if:

– it is dissipative: $|det(D_x f)| < 1$ for all $x \in \mathbb{D}$,

– $\forall$ ergodic measure $\mu$ ($\neq$ sink), and $\mu$-a.e. $x$, $W^s(x)$ separates $\mathbb{D}$.



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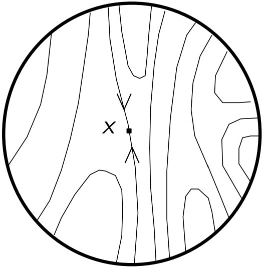
Examples.

– Hénon maps for $|b| < 1/4$:
$(x, y) \mapsto (1 - ax^2 + y, -bx)$
(uses Wiman's theorem).

– Maps close to 1D endomorphisms.




Property. (C - Pujals)

– When $|det(Df)|$ is small enough, strong dissipation is $C^2$-open.

– *Closing.* The set of periodic points is dense in the supports of invariant probabilities.

# Renormalization

**Theorem A.** (C-Pujals-Tresser) *For any strongly dissipative diffeomorphism $f$ of the disc whose topological entropy vanishes,*

- a- *either any forward orbit of $f$ converges to a fixed point,*
- b- *or $f$ is renormalizable: there exists a topological disc $U$ and $m \geq 2$ such that $f^m(U) \subset U$ and $f^i(U) \cap U = \emptyset$ when $0 < i < m$.*

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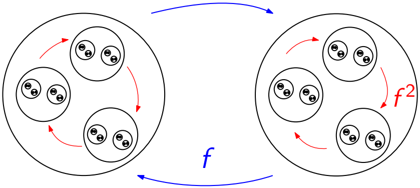
In case (b), the result also applies to the renormalization $f^m|_U$.

$f$ is *infinitely renormalizable* when there exists an infinite sequence of successive renormalizations.



# Decomposition of the dynamics: periodic structure

**Theorem A.** *For $f$ strongly dissipative with $h_{top}(f) = 0$, there is $W$:*

*(1) $W$ (maybe $\emptyset$) is the finite union of disjoint renormalization domains:*

$$U_i \cup f(U_i) \cup \dots \cup f^{m_i - 1}(U_i), 1 \leq i \leq \ell,$$

*which are trapped: $f^{m_i}(\overline{U}_i) \subset interior(U_i)$,*

*(2) in $\mathbb{D} \setminus W$, the dynamics is (generalized) Morse-Smale:*

- *any forward orbit of $f$ in $\mathbb{D} \setminus W$ converges to a periodic point,*
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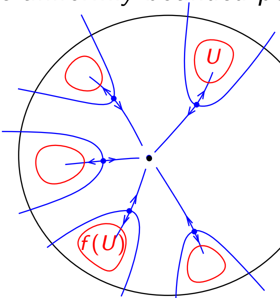
The renormalization domains are "attached" to particular saddle orbits that are "decorated" around a fixed point.



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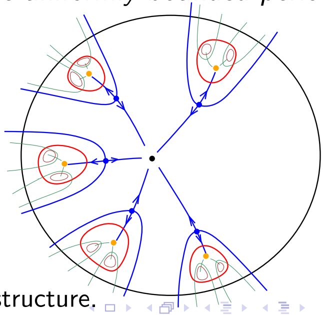


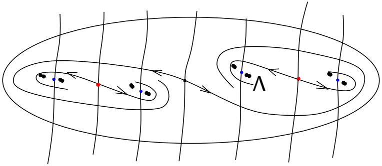
$\Rightarrow$ The set of periodic orbits has a hierarchical structure.

# Decomposition of the dynamics: aperiodic structure

When $f$ is infinitely renormalizable: $(W_k)$: nested sequence of renormalization loci, decreases to an aperiodic set $\Lambda$.

**Corollary.** *For $f$ strongly dissipative with $h_{top}(f) = 0$, there exists an invariant compact set $\Lambda$ such that:*

- *$\Lambda$ is the disjoint union of invariant compact sets $K_i$,*
- *any forward orbit accumulates on a periodic orbit or on some $K_i$,*
- *the periodic orbits of large periods are close to $\Lambda$,*
- *each $K_i$ is semi-conjugated to an odometer: $\pi_i \colon K_i \to \mathcal{O}_i$, for a.e. $x \in \mathcal{O}_i$, $\pi_i^{-1}(x)$ is a singleton. (So $K_i$ is uniq. ergodic.)*




$\Rightarrow$ *This describes all the chain-transitive sets.*

# Gambaudo-Tresser conjecture

$f$: a strongly dissipative diffeomorphism of the disc with
– zero entropy,
– periodic points with arbitrarily large period.

**Theorem B.** *There exist* $W$ *open and* $m \geq 1$ *such that* $f(\overline{W}) \subset W$ *and:*
– *the periods of the periodic points of* $f^m$ *in* $W$ *are* $\{2^n, n \in \mathbb{N}\}$,
– *the periods of the periodic points in* $\mathbb{D} \setminus W$ *is bounded.*

*Corollary. There exist integers* $m_1, \dots, m_\ell$ *such that the set of periods of the periodic orbits of* $f$ *coincides with*

$$Per(f) = \{m_i.2^n, \;\; n \geq 0, 1 \leq i \leq \ell\} \cup \textit{finite set}.$$

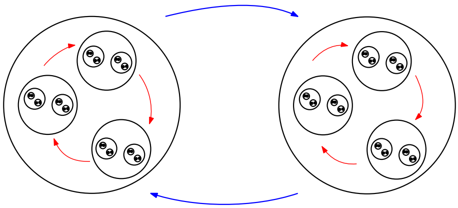
# Gambaudo-Tresser conjecture: thm A $\Rightarrow$ thm B

Consider a nested sequence of renormalizations domains

$$U_k \cup f(U_k) \cup \cdots \cup f^{m_k - 1}(U_k) \text{ with period } m_k \to +\infty,$$

and decreasing towards an odometer $K$.

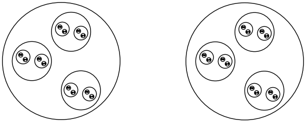
Goal. *When $m_k$ is large, $m_{k+1} = 2m_k$.*



$\Rightarrow$ All periodic points in $U_k$ have periods in $\{m_k.2^n : n \geq 0\}$.

# Gambaudo-Tresser conjecture: $\gamma$-dissipation

$K$: a limit odometer with trapping neighborhood $U$.



Observation. *The dynamics on $U$ is 9/10-dissipative:*

$$\text{for } x, y \in U,\ u \in T^1_y M, \qquad |det(D_x f)| \le \|D_y f.u\|^{9/10}.$$

Indeed: the (aperiodic) measure on $K$ can not be hyperbolic since $h_{top} = 0$. Hence its maximal Lyapunov exponent vanishes.

Quantitative Pesin theory. *If $f$ is 9/10-dissipative, there is a set $X$ s.t.*

– *$W^s(x)$ varies continuously with $x \in X$ for the $C^1$-topology,*

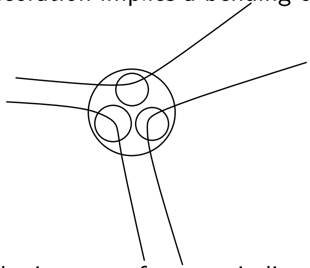
– *$\mu(X) > 2/3$ for any measure $\mu$ supported on $U$.*

# Gambaudo-Tresser conjecture: no decoration

$K$: a limit odometer with trapping neighborhoods

$U_k \cup f(U_k) \cup \cdots \cup f^{m_k-1}(U_k)$ with period $m_k \to +\infty$,

*When* $\frac{m_{k+1}}{m_k} \geq 3$, the decoration implies a bending of the stable manifolds..

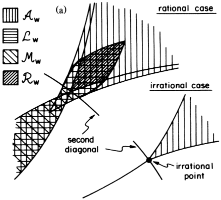


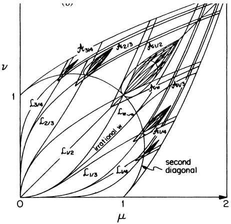
... for at least $1/3$ of the iterates of any periodic orbit close to $K$.

This contradicts the uniform Pesin theory!

Conclusion: $\frac{m_{k+1}}{m_k} = 2$ for $k$ large.

This gives thm B (Gambaudo-Tresser conjecture) from thm A (renormalization).




**Dear JMGA, my little brother** (and hello! to the big brother whose 65th I also missed).
**Together we have explored many pieces of the boundary of topological chaos, among many math destinations that we took together. Then life and my quasi deaths allowed an edge so wide that I do not even really know your kids (despite a paper with your wife. Sorry to not be able to** make it to your 6oth conference: at least you will not fill the need tell me to stay away because I ask too many questions (but others, more mighty than I am, will avenge me!).

Till very recently I was hopping to be part of the FIESTA but some circumstances went in the way (Inferno is paved with Nice's unpredictability !).

For reasons you know, I know that it his NOT your birthday (by 6 months).

THUS I wish you a happy celebration of your mathematical work

(And I thank Sylvain for transmitting this, & hughs –and-kisses to Betty and you).

# MAZEL TOV!

(and when do we dive into our next math paper?).