Eigenvalues and strong orbit equivalence

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General Context

- We deal with **minimal Cantor systems**, i.e, dynamical systems (X, T) where X is a Cantor set and $T : X \to X$ is a minimal homeomorphism.
- Minimal Cantor systems can be classified in terms of **orbit equivalence** (O.E) classes and **strong orbit equivalence** (S.O.E) classes.
- Some properties of minimal Cantor systems are invariant under O.E and S.O.E. **What about the group of eigenvalues?**
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Orbit equivalence

- (X, T) and (Y, S) are **orbit equivalent** if there exists a homeomorphism $h : X \to Y$ such that for every $x \in X$, $$h(O_T(x)) = O_S(h(x)).$$
- Define $n, m : X \to \mathbb{Z}$ as $h(T(x)) = S^{n(x)}(h(x))$ and $h(T^{m(x)}(x)) = S(h(x))$. The systems are **strong orbit equivalent** if n and m have at most one point of discontinuity.
- (Boyle) n and m are continuous iff the systems are **flip conjugate** (T is conjugate to S or S^{-1}).
- Conjugacy $\Rightarrow$ flip conjugacy $\Rightarrow$ S.O.E $\Rightarrow$ O.E.

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Algebraic invariants

(Giordano, Putnam, Skau 95, 96)

Property	Algebraic Invariant
Flip conjugacy	topological full group
Strong orbit equivalence	**dimension group**
Orbit equivalence	reduced dimension group and full group

Properties which are invariant

- If (X, T) and (Y, S) are O.E then there is an affine bijection between their spaces of invariant probability measures.
- Thus systems with different numbers of ergodic measures can not be O.E.
- The converse is not true (there are infinitely many S.O.E classes among the family of uniquely ergodic systems).
- The topological entropy fails: for every $t \in [0, \infty]$ and for every minimal Cantor system (X, T), there exists a minimal Cantor system (Y, S) which is S.O.E with (X, T) and such that $h_{top}(Y, S) = t$.
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Eigenvalues

- Let (X, T) be a minimal Cantor system.
- An **eigenvalue** of (X, T) is a number $\alpha \in \mathbb{R}$ for which there exists $f \in C(X, \mathbb{C}) \setminus \{0\}$ (called eigenfunction) such that
$$f \circ T = \exp(2i\pi\alpha)f.$$
- If T is transitive (in particular, minimal) and $f \in C(X, \mathbb{C})$ is an eigenfunction associated to the eigenvalue α, then $|f|$ is constant. Normalizing f we can assume that $f : X \to S^1$.
- Thus f is a factor map from (X, T) to (S^1, R_α), where R_α is the α rotation on the circle.

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Group of eigenvalues

- We denote $E(X, T)$ the subset of $\mathbb{R}$ containing all the eigenvalues of (X, T).
- **Remark:** $E(X, T)$ is a subgroup of $(\mathbb{R}, +)$.
- **Remark:** $\mathbb{Z} \subseteq E(X, T)$ (the constant functions are eigenfunctions associated to the integers).
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Dimension group

- Let (X, T) be a minimal Cantor system.
- Consider the additive group $C(X, \mathbb{Z})$ (this is abelian and countable).
- Consider the subgroup $\partial_T C(X, \mathbb{Z}) = \{f - f \circ T : f \in C(X, \mathbb{Z})\}$.
- We define $D(X, T) = C(X, \mathbb{Z}) / \partial_T C(X, \mathbb{Z})$ and $D^+(X, T) = \{[f] : f \geq 0\}$.
- The pair $(D(X, T), D^+(X, T))$ is an ordered group.
- The **dimension group with unit** associated to (X, T) is $K^0(X, T) = (D(X, T), D^+(X, T), [1_X])$.

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Invariant for strong orbit equivalence.

Theorem

(Giordano, Putnam, Skau 95) (X, T) *and* (Y, S) *are S.O.E if and only if* $K^0(X, T)$ *and* $K^0(Y, S)$ *are isomorphic as dimension groups with unit.*

States or traces

- A **state** or **trace** of $K^0(X, T)$ is a homomorphism $\tau : D(X, T) \to \mathbb{R}$ such that $\tau(D^+(X, T)) \subseteq \mathbb{R}^+$ and $\tau([1_X]) = 1$.
- It is not difficult to show that the following function is an affine bijection between $\mathcal{M}(X, T)$ and the set of traces:

$$\mu \longrightarrow \tau_\mu \text{ where } \tau_\mu([f]) = \int f d\mu.$$

- (Many ways to show this):

$$E(X, T) \subseteq I(X, T) = \bigcap_{\tau \text{ trace}} \tau(D(X, T)).$$

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Example

- Let $\alpha \in [0,1] \setminus \mathbb{Q}$ and (X_α, T_α) be the associated Sturmian subshift.
- $K^0(X_\alpha, T_\alpha)$ is isomorphic to $(\mathbb{Z} + \alpha\mathbb{Z}, (\mathbb{Z} + \alpha\mathbb{Z}) \cap \mathbb{R}^+, 1)$.
- The only trace for this dimension group is the identity. Then $I(X_\alpha, T_\alpha) = \mathbb{Z} + \alpha\mathbb{Z}$.
- On the other hand, we know that $\alpha \in E(X_\alpha, T_\alpha)$, then $\mathbb{Z} + \alpha\mathbb{Z} \subseteq E(X_\alpha, T_\alpha)$
- We deduce $\mathbb{Z} + \alpha\mathbb{Z} = E(X_\alpha, T_\alpha) = I(X_\alpha, T_\alpha)$.
- If (X, T) is S.O.E with (X_α, T_α) then $E(X, T)$ is a subgroup of $\mathbb{Z} + \alpha\mathbb{Z}$. What kind of subgroups of $\mathbb{Z} + \alpha\mathbb{Z}$ we can realize as $E(X, T)$?

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Rational subgroup

- The **rational group** associated to $K^0(X,T)$ is defined as

$$Q(K^0(X,T)) = \langle\{\frac{1}{n} : n \in \mathbb{N} \text{ s.t } \exists[f] \in D(X,T) \text{ s.t } n[f] = [1_X]\}\rangle.$$

- $\mathbb{Z} \leq Q(K^0(X,T)) \leq \mathbb{Q}$.
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Rational eigenvalues and rational subgroup

Ormes 97:

- $E(X,T) \cap \mathbb{Q} = Q(K^0(X,T))$.
- Thus the rational eigenvalues are invariant under S.O.E.
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- The **infinitesimal subgroup** of $K^0(X,T)$ is

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Eigenvalues and dimension groups.

Theorem

(C., Durand, Petite, 2016) Let (X, T) be a minimal Cantor system such that the infinitesimal subgroup of $K^0(X, T)$ is trivial. Then $I(X, T)/E(X, T)$ is torsion free or trivial. This is not always true if the infinitesimal subgroup is not trivial.

Example and questions

- If $K^0(X, T)$ is isomorphic to $K^0(X_\alpha, T_\alpha)$ then $E(X, T) \in \{\mathbb{Z}, \mathbb{Z} + \alpha\mathbb{Z}\}$.
- We know that each group in $\{\mathbb{Z}, \mathbb{Z} + \alpha\mathbb{Z}\}$ is realized as the group of eigenvalues of a system in the S.O.E class of the Sturmian subshift.
- What about the groups of eigenvalues that can be realized in any other S.O.E class?
- Suppose that α and β are rationally independent. If $K^0(X, T)$ is isomorphic to $\mathbb{Z} + \alpha\mathbb{Z} + \beta\mathbb{Z}$, then $E(X, T)$ is in $\{\mathbb{Z}, \mathbb{Z} + \alpha\mathbb{Z}, \mathbb{Z} + \beta\mathbb{Z}, \mathbb{Z} + \alpha\mathbb{Z} + \beta\mathbb{Z}\}$.
- In the previous case, it is possible to realize $\mathbb{Z}$, $\mathbb{Z} + \alpha\mathbb{Z}$ and $\mathbb{Z} + \beta\mathbb{Z}$ as $E(X, T)$. What about $\mathbb{Z} + \alpha\mathbb{Z} + \beta\mathbb{Z}$?

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(Giordano, Handelman, Hosseini, 2016) $K^0(X,T)/\theta(E(X,T))$ *is torsion free.*

- The dimension group is irrationally miscible if $I(X,T) \subseteq \mathbb{Q}$.
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Feliz Cumple!!!