Tilings, Towers and Linear repetitivity

(On Jean-Marc's relationship with Chile)

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Main motivation

- Let (Ω, T) be the dynamical system associated with a non-periodic tiling or Delone set. Can we understand/compute its ergodic/mixing properties? What about its eigenvalues?

Some History: Minimal Cantor Systems

- A minimal Cantor system is a pair (X, T) where X is a Cantor set, $T : X \to X$ is a *minimal* homeomorphism, i.e., every orbit is dense in X.
- λ in $\mathbb{S}^1$ is an eigenvalue of (X, T) if there exists $f \in L^2(X, \mathbb{S}^1)$ s.t.

$$f(Tx) = \lambda f(x) \quad \text{a. e.}$$

- If f is continuous and the latter equation holds everywhere, the eigenvalue is *continuous*.
- (X, T) is (topol.)*weakly mixing* if there are no non-trivial (continuous) eigenvalues.

Kakutani Rohlin partitions of minimal Cantor sets

- A *Rohlin tower* of (X, T) is a family of pairwise disjoint measurable sets of the form

$$\{C, T^{-1}C, \dots T^{-h}C\}$$

- A Kakutani-Rohlin partition is a partition that is a finite union of disjoint Rohlin Towers.

Kakutani-Rohlin partitions: (incomplete) history

- In the 40's, Kakutani-Rohlin towers were introduced in the ergodic setting by Kakutani and Rohlin (independently).
- In 1992, Hermann, Putnam and Skau introduced clopen KR partitions in the context of minimal Cantor systems.
- In 1999. Durand, Host and Skau used KR partitions to understand substitution dynamical systems.
- In 2003. Cortez, Durand, Host and Maass gave a first characterization of eigenvalues for linearly recurrent cantor systems.
- In 2000? Gambaudo and Martens described Minimal Cantor systems as inverse limits that are parallel to Kakutani Rohlin partitions .
- In 2000? Gambaudo, Beneddetti and Bellissard using tower systems proved the Gap Labelling conjecture.

Kakutani-Rohlin partitions: (The Chile papers)

- Jean-Marc and Elisabeth move to Chile and bring along Samuel with them
- Gambaudo, Guiraud and Petite studied Frenkel Kontorova models on 1D-quasicrystals by using Kakutani-Rohlin partitions.
- Cortez, Gambaudo and Maass characterized rotation factors for actions of $\mathbb{Z}^d$ over the Cantor set.
- Jean-Marc and Elisabeth move back to France and bring along Daniel, Guillermo and José with them.

Alejandro's and Jean-Marc's pictures for the towers of Fibonacci

- Draw picture here.

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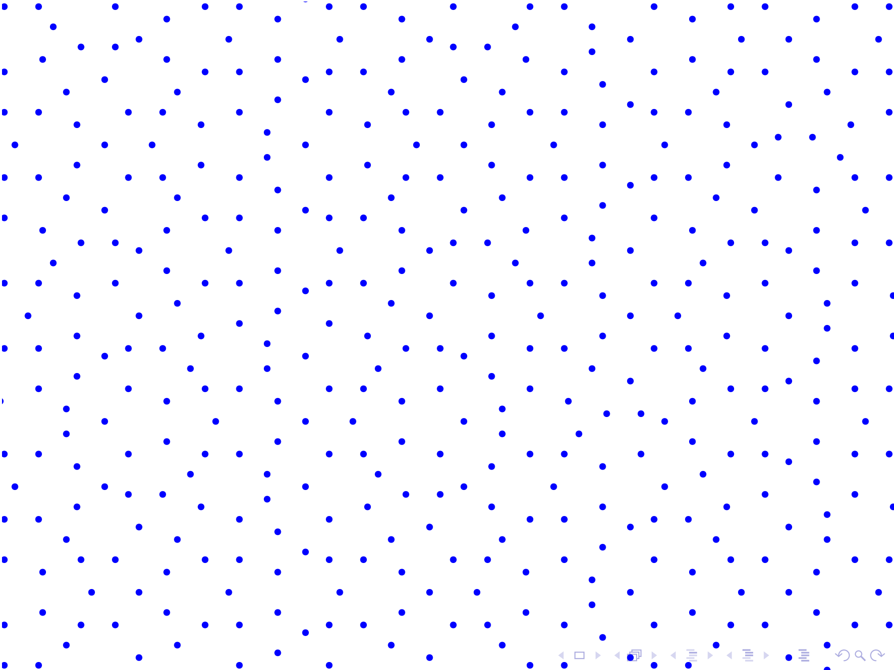
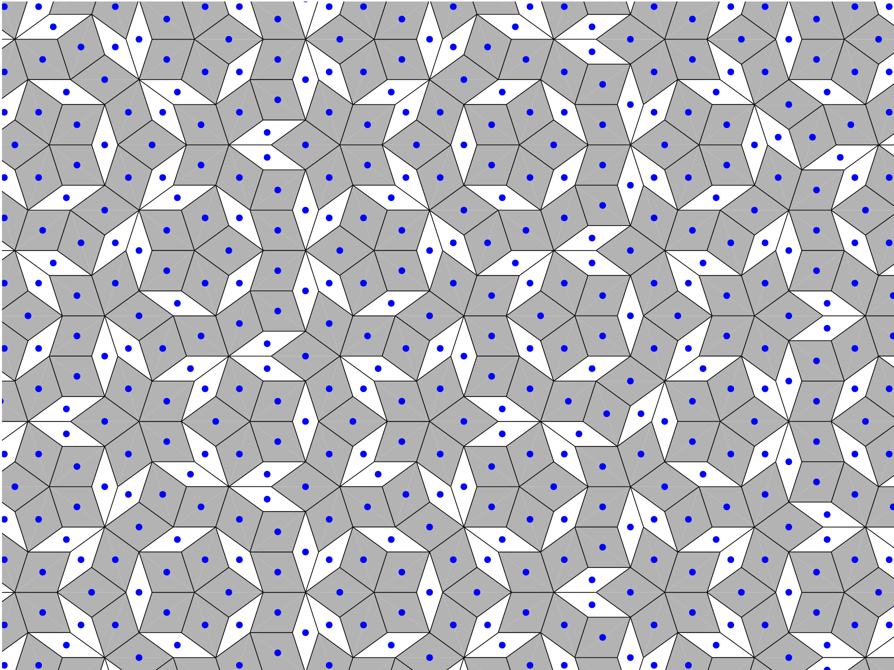
Delone sets



Delone sets



Repetitive and Linearly Repetitive Delone sets

- A R-patch of X is the configuration of points in a ball of radius R centered at a point of X. A Delone set X is repetitive if for each patch P of X there is a radius M such that every ball of radius M contains the center of a translated copy of P.
- If $M \leq LR$, for some constant L then X is linearly repetitive.
- Some substitutions and cut-and-project Delone sets are linearly repetitive.

Delone dynamical systems

Let X be a repetitive Delone set. The Delone dynamical system is defined by

$$\Omega = \{X - t \mid t \in \mathbb{R}^d\}$$

endowed with a ”good” topology and where $\mathbb{R}^d$ acts on Ω by translations.
If we only consider Delone sets such that 0 belongs to the set, then we get a Cantor set, called the transversal.

Theorem (???)

Ω *has the structure of a lamination with Cantor transversal.*

Theorem (Gambaudo, Bennedetti, Bellissard)

Ω *has the structure of a flat lamination.*

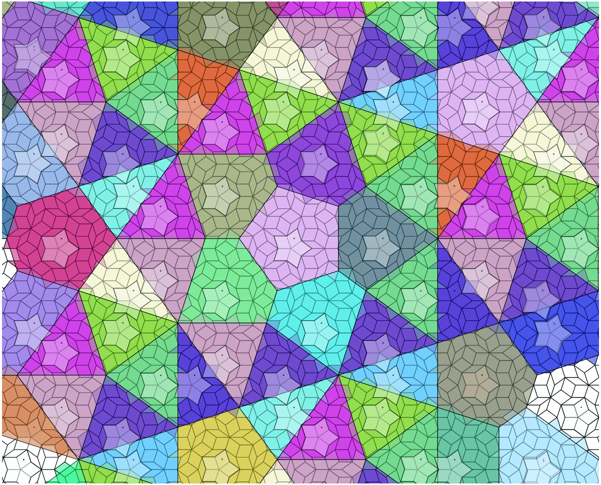
Box decompositions and tower systems

- Kakutani Rohlin partitions can be generalized to box decompositions. A Box is a set B in Ω homeomorphic to $C \times D$ where C is a Cantor set and D is a closed disc (or a polygon).
- a Box decomposition is a cover of Ω by boxes whose interiors are pairwise disjoints.
- a Tower system is a sequence of box decompositions that are ”transversally finer” and ” horizontally finer”

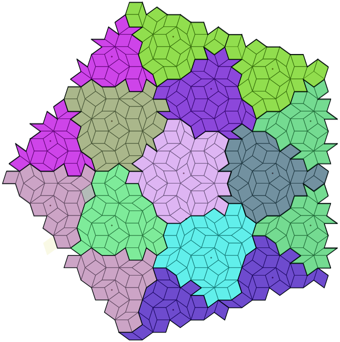
Derived tilings and Tower systems

- If we intersect a box with the orbit of a Delone set X in Ω, by drawing the boundary of the box on the orbit (which is $\mathbb{R}^d$), and doing the same for all the boxes in a box decomposition, we get a tiling of $\mathbb{R}^d$ which is locally derived from X.
- transverally "Finer" for tower systems mean that the transversals of the finer box decomposition are included in the transversals of the coarser box, and horizontally finer means that the derived tilings of the finer box decomposition are supertilings of the coarser box decomposition.

Derived tilings from a box decomposition



Making a box decompositon horizontally finer than another



Tower systems or Delone dynamical systems as inverse limits

- From a Box decomposition, by collapsing the transversal one gets a branched manifold with flat structure (Gambaudo, Bennedetti, Bellisard).
- From this one deduces that the Delone system can be viewed (geometrically) as the inverse limit of branched flat manifolds.

Tower systems for linearly repetitive systems

- The main idea of linearly repetitive is that one has uniform bound on the number of boxes in each box decomposition in a tower system. This allows for bounding the growth of the size of the faces in the branched manifolds as well as the number of faces.
- With this we give a new proof for a well-known theorem of Lagarias Pleasants that establishes the unique ergodicity of linearly repetitive DElone systems and also give some bounds for the rate of convergence.

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Cut and Project scheme

A *cut and project scheme* (CPS) is the data of

- a locally compact Abelian group H.
- a lattices $\widetilde{L}$ in $\mathbb{R}^d \times H$ such that
  - its projection on $\mathbb{R}^d$ is one-to-one and
  - its projection on H is dense.
- a compact subset W of H that is the closure of its interior.

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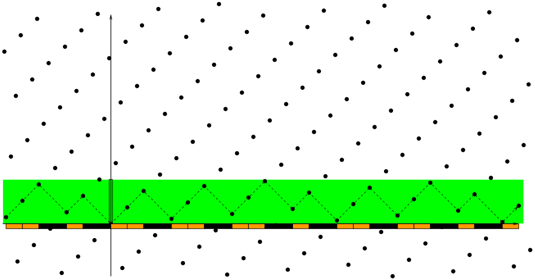
  - its projection on H is dense.
- a compact subset W of H that is the closure of its interior.

The locally compact Abelian group H is called *the internal space* and the set W is called *the window*.

Cut and Project set or model set

From a CPS $(H, \widetilde{L}, W)$ one define a *cut and project set or model set* Λ by projecting the intersection $\widetilde{L} \cap \mathbb{R}^d \times W$ on $\mathbb{R}^d$.

If $\text{meas}(\partial W) = 0$ we say that the model set is *regular*.



Dynamical eigenvalues

- Assume that $(X, \mathbb{R}^d, \mu)$ is an ergodic dynamical system.
- k in $\mathbb{R}^d$ is an eigenvalue of $(X, \mathbb{R}^d, \mu)$ if and only if there is f in $L^2(X, \mu)$ with $f \neq 0$ such that for μ almost every Λ' in X and for every t in $\mathbb{R}^d$,

  $$f(\Lambda' - t) = \exp(i2\pi k \cdot t) f(\Lambda').$$

  f is called the eigenfunction to k.
- An eigenvalue is continuous or topological if it has a continuous eigenfunction.

Cut and Project scheme and model sets

Theorem (Hof 1995, Schlottmann 2000)

The hull of every regular model set is uniquely ergodic and pure point. Moreover, all the eigenvalues are topological and the maximal equicontinuous factor is $(\mathbb{T} := (\mathbb{R}^d \times H)/\widetilde{L}, \mathbb{R}^d)$.

Meyer sets

Definition

A Delone set Λ is a *Meyer set* if and only if there is a finite set F such that

$$\Lambda - \Lambda \subset \Lambda + F.$$

Examples: $\mathbb{Z}^d$, lattices, Delone sets in lattices, some substitutions, model sets.

Theorem (Lagarias 1995)

Λ is a Meyer set if and only if $\Lambda - \Lambda$ is a Delone set.

Charaterization of regular model sets

Theorem (Baake - Lenz - Moody 2006)

Let X be the hull of a repetitive Delone set Λ with finite local complexity. The following are equivalent:

(i) *X is the hull of a regular model set.*

(ii) *Λ is a Meyer set and the set of points in its maximal equicontinuous factor with a unique preimage under the factor map has full measure.*

Charaterization of model sets

Theorem (Aujogue 2014)

Let X be the hull of a repetitive Delone set Λ with finite local complexity. The following are equivalent:

(i) *X is the hull of a model set.*

(ii) *Λ is a Meyer set and there is a point in its maximal equicontinuous factor with a unique preimage under the factor map.*

Address map

- Let Λ be a FLC Delone set in $\mathbb{R}^d$ with $0 \in \Lambda$.
- The Abelian group $\Gamma := \langle \Lambda - \Lambda \rangle$ is finitely generated.
- We say that Λ has *rank* r if Γ has rank r.
- Let $\{v_1, \ldots, v_r\}$ be a basis for $\langle \Lambda - \Lambda \rangle$.

Address map

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- Let $\{v_1, \ldots, v_r\}$ be a basis for $\langle \Lambda - \Lambda \rangle$.

The *address map* $\varphi : \langle \Lambda - \Lambda \rangle \to \mathbb{Z}^r$,

$$x = \sum_{j=1}^{r} n_j v_j \in \langle \Lambda - \Lambda \rangle \mapsto (n_1, \ldots, n_r)^T.$$

Address map

Theorem (Lagarias 1999)

A Delone set Λ *in* $\mathbb{R}^d$, *with* $0 \in \Lambda$, *is a Meyer set if and only if it has finite local complexity and the address map*

$$\varphi : \langle \Lambda - \Lambda \rangle \to \mathbb{Z}^r$$

is almost linear, that is, there is a linear map $\ell : \mathbb{R}^d \to \mathbb{R}^r$ *and a constant* $C > 0$ *such that*

$$\forall x \in \Lambda, \quad \|\varphi(x) - \ell(x)\| \leq C.$$

Theorem (Allendes-C.)

For every repetitive Meyer set Λ *of rank* r *with* $0 \in \Lambda$*, and every address map* φ *of* Λ*, the linear map* ℓ *approximating* φ *gives* $r \geq d$ *continuous eigenvalues, all of them* $\mathbb{Z}$*-linearly independent.*

Theorem (Allendes-C.)

For every repetitive Meyer set Λ of rank r with $0 \in \Lambda$, and every address map φ of Λ, the linear map ℓ approximating φ gives $r \geq d$ continuous eigenvalues, all of them $\mathbb{Z}$-linearly independent.

More precisely, the columns of the representative matrix of ℓ in the canonical bases are continuous eigenvalues.

Topological essential diffractivity

Theorem (Kellendonk - Sadun 2012)

A repetitive Delone set in $\mathbb{R}^d$ with local finite complexity has d linearly independent topological eigenvalues if and only if it is topologically conjugate to a repetitive Meyer set.

In particular, every repetitive Meyer set has d linearly independent topological eigenvalues.

Charaterization of model sets

Theorem (Allendes-C.)

Let X be the hull of a repetitive aperiodic Delone set Λ with finite local complexity. The following are equivalent:

(i) *X is the hull of a model set with Euclidean internal space.*

(ii) *Λ is a Meyer set with rank $r > d$, there is a point in its maximal equicontinuous factor with a unique preimage under the factor map, and the maximal equicontinuous factor is topologically conjugate to the factor induced by the address map.*

Charaterization of regular model sets

Theorem (Allendes-C.)

Let X be the hull of a repetitive aperiodic Delone set Λ with finite local complexity. The following are equivalent:

(i) *X is the hull of a regular model set.*

(ii) *Λ is a Meyer set with rank $r > d$, the set of points in its maximal equicontinuous factor with a unique preimage under the factor map has full measure, and the maximal equicontinuous factor is topologically conjugate to the factor induced by the address map.*

Feliz Cumpleaños Jean-Marc

The transverse space

Let X be the hull of a repetitive aperiodic Delone set. The transversal of X is the set

$$\Xi := \{\Lambda \in X : 0 \in \Lambda\}.$$

It is a Cantor set.

The transverse groupoid

Consider the set

$$\mathcal{G} := \{(\Lambda, t) \in \Xi \times \mathbb{R} : \Lambda - t \in \Xi\}.$$

Two elements (Λ_1, t_1) and (Λ_2, t_2) in $\mathcal{G}$ are composable if $\Lambda_2 = \Lambda_1 - t_1$, in this case we define

$$(\Lambda_1, t_1) \cdot (\Lambda_2, t_2) := (\Lambda_1, t_1 + t_2).$$

With the induced topology and this local operation $\mathcal{G}$ is a topological groupoid.

A continuous and bounded cocycle

Since the dynamical system $(X, \mathbb{R}^d)$ is minimal the groupoid $\mathcal{G}$ is also minimal.

For every Λ in Ξ the group $\langle \Lambda - \Lambda \rangle$ is independent of Λ. Choose a basis for this group and define all the address maps $(\varphi_\Lambda)_{\Lambda \in \Xi}$ with this basis.

By Lagarias Theorem for every Λ in Ξ there is a linear map ℓ_Λ and a constant C_Λ such that for every (Λ, t) in $\mathcal{G}$ we have

$$\|\varphi_\Lambda(t) - \ell_\Lambda(t)\| \leq C_\Lambda.$$

A continuous and bounded cocycle

Lemma

The map ℓ_Λ is independent of Λ and we denoted it by ℓ.

For every (Λ, t) in $\mathcal{G}$ put

$$\Phi(\Lambda, t) = \phi_\Lambda(t) - \ell(t).$$

Φ is a cocycle on $\mathcal{G}$: given two composable elements (Λ_1, t_1) and (Λ_2, t_2) we have

$$\Phi((\Lambda_1, t_1) \cdot (\Lambda_2, t_2)) = \Phi(\Lambda_1, t_1) + \Phi(\Lambda_2, t_2).$$

Gottchalk-Hedlund Theorem

Hypotheses of Gottchalk-Hedlund Theorem:

- The groupoid $\mathcal{G}$ is minimal.
- The cocycle Φ is continuous.
- There is Λ_0 in Ξ such that Φ is bounded on the orbit of Λ_0: $\{\Phi(\Lambda_0, t) : \text{ for every } t \in \Lambda_0\}$ is bounded.

Conclusion: There is a continuous map $F : \Xi \to \mathbb{R}^r$ such that for every (Λ, t) in $\mathcal{G}$ we have

$$\Phi(\Lambda, t) = F(\Lambda) - F(\Lambda - t).$$

Topological eigenvalues

In each coordinate we have

$$\varphi(\Lambda, t)_j - \ell(t)_j = F(\Lambda)_j - F(\Lambda - t)_j.$$

Then,

$$e^{-i2\pi\ell(t)_j} = e^{i2\pi F(\Lambda)_j} e^{-i2\pi F(\Lambda - t)_j}.$$

Putting $f_j(\Lambda) = e^{i2\pi F(\Lambda)_j}$ we get

$$f_j(\Lambda - t) = e^{i2\pi\ell(t)_j} f_j(\Lambda), \text{ for every } t \in \Lambda.$$