Numerical computation of invariant objects with wavelets

Lluís Alsedà

in collaboration with D. Romero

Centre de Recerca Matemàtica

Departament de Matemàtiques
Universitat Autònoma de Barcelona

http://www.mat.uab.cat/∼alseda

Systèmes dynamiques et systèmes complexes Nice, June 14th, 2018

Outline

Motivation

We are interested in studying complicate objects semianalitically (obtaining expansions in a truncated base) to be able to predict and understand changes in the geometry or dynamical properties as reducibility and others.

To fix ideas let us describe a couple of models that we can use as *toy models*.

We consider skew products of the form

$$\mathfrak{F}_{\sigma,\varepsilon}\begin{pmatrix}\theta_n\\ x_n\end{pmatrix}=\begin{cases}\theta_{n+1} &= R(\theta_n)=\theta_n+\omega \pmod 1,\\ x_{n+1} &= T(\theta,x)\end{cases} \tag{1}$$

where $x\in\mathbb{R}^+, \theta\in\mathbb{S}^1$, $\omega\in\mathbb{R}\setminus\mathbb{Q}$ and $T(\theta,x)$ is of the form either $f(x)g(\theta)$ or $f(x)+g(\theta)$.

The [GOPY]-Keller model

In the system (1) we take $T(\theta, x) = f(x)g(\theta)$ (*multiplicative forcing*) with

1. $f : [0, \infty) \longrightarrow [0, \infty) \in \mathcal{C}^1$, bounded, strictly increasing, strictly concave and verifying $f(0) = 0$ (to fix ideas take $f(x) = 2\sigma \tanh(x)$ with $\sigma > 0$ as in the [GOPY] model). Thus, $x = 0$ will be invariant.
2. $g : \mathbb{S}^1 \longrightarrow [0, \infty)$ bounded and continuous (to fix ideas take $g(\theta) = \varepsilon + |\cos(2\pi\theta)|$ in a similar way to the [GOPY] model – except for ε and the absolute value).

We get:

$$\mathfrak{F}_{\sigma,\varepsilon} \begin{pmatrix} \theta_n \\ x_n \end{pmatrix} = \begin{cases} \theta_{n+1} & = \theta_n + \omega \pmod 1, \\ x_{n+1} & = 2\sigma \tanh(x_n)(\varepsilon + |\cos(2\pi\theta_n)|) \end{cases} \tag{2}$$

$\omega = \frac{\sqrt{5}+1}{2}$, $\sigma > 0$ and $\varepsilon \geq 0$.

[GOPY] Grebogi, Celso *et al.*, *Strange attractors that are not chaotic*, Phys. D 13 1984 1–2 261–268.

The [GOPY]-Keller model

We want to approximate the attractor of the above system (if it exists).

Pinching condition $\Rightarrow$ SNA's creation

When $g = 0$ at some point it is called the *pinched case*, whereas if g is strictly positive it is called the *non-pinched case*.

In the pinched case, any $\mathfrak{F}_{\sigma,\varepsilon}$–invariant set has to be 0 *on a point* and, hence, *on a dense set* (in fact on a *residual set*). This is because the circle $x \equiv 0$ is invariant and the θ-projection of every invariant object must be invariant under R_ω.

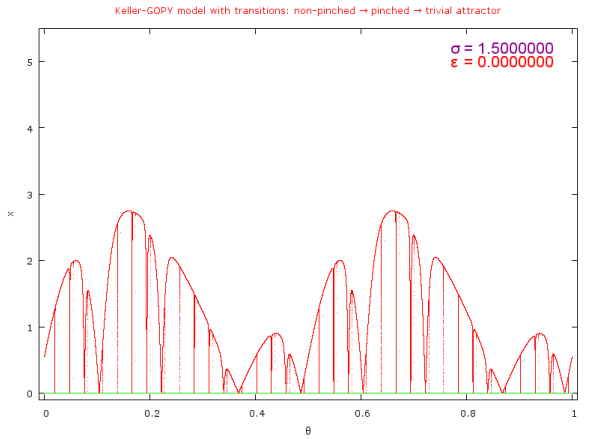
The [GOPY]-Keller model

The following theorem due to Keller [Kel] clarifies the dynamics of these models. Before stating it we need to clarify the rôle of the constant σ:

Since the line $x = 0$ is invariant, by using Birkhoff Ergodic Theorem, it turns out that

$$\sigma := f'(0) \exp\left(\int_{\mathbb{S}^1} \log g(\theta) d\theta \right) < \infty.$$

is the vertical Lyapunov exponent on the circle $x = 0$.



▸ Movie: A family of Keller Attractors

[Kel] Keller, Gerhard, *A note on strange non-chaotic attractors*, Fund. Math. 151 1996 2 139–148.

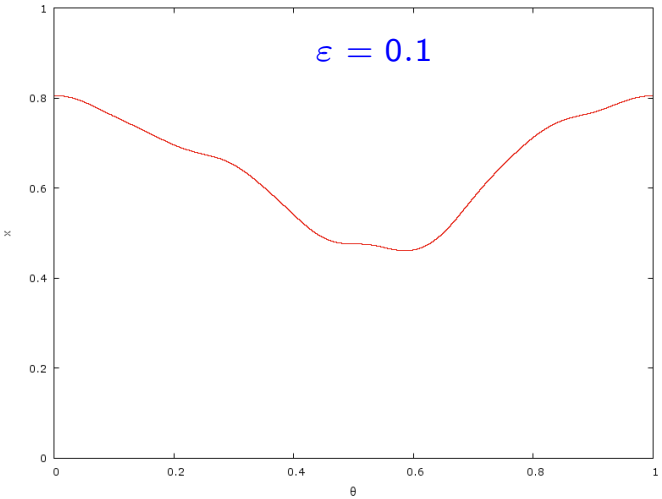
Keller Theorem

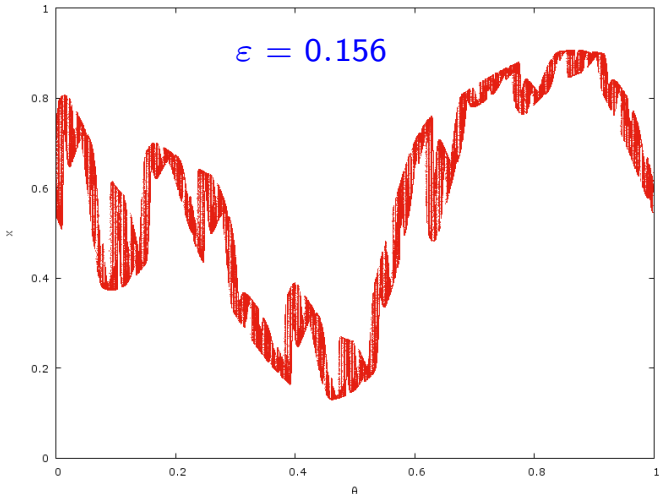
There exists an upper semicontinuous map $\phi\colon \mathbb{S}^1 \longrightarrow [0,\infty)$ whose graph is invariant under the Model (2). Moreover,

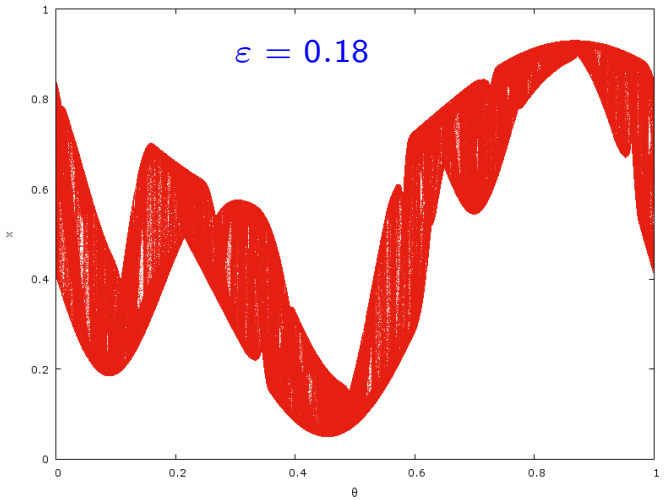
1. The Lebesgue measure on the circle, lifted to the graph of ϕ is a Sinai-Ruelle-Bowen measure,
2. if $\sigma \leq 1$ then $\phi \equiv 0$,
3. if $\sigma > 1$ then $\phi(\theta) > 0$ for almost every θ,
4. if $\sigma > 1$ and $g(\theta_0) = 0$ for some θ_0 then the set $\{\theta\colon \phi(\theta) > 0\}$ is meager and ϕ is almost everywhere discontinuous,
5. if $\sigma > 1$ and $g > 0$ then ϕ is positive and continuous; if g is $\mathcal{C}^1$ then so is ϕ,
6. if $\sigma \neq 1$ then $|x_n - \phi(\theta_n)| \to 0$ exponentially fast for almost every θ and every $x > 0$.

For this model we want to compute the attractor so that we can detect the pinching parameter (regularity).

The Nishikawa-Kaneko model
Three different plots for $a = 3.0$







$$\begin{cases} \theta_{n+1} &= \theta_n + \omega \pmod 1, \\ x_{n+1} &= ax(1-x) + \varepsilon \sin(2\pi\theta_n) \end{cases} \qquad (3)$$

where $x \in [0,1]$ and $\omega = \frac{\sqrt{5}+1}{2}$.

The Nishikawa-Kaneko model

In the Nishikawa-Kaneko paper it is described the *fractalization route* (that they illustrate in the pictures) with support on rough numerical computations.

Currently there is a strong debate about this route and about the fact that what we get at the end is really a fractal.

For this model we want to compute the attractors so that we can approximate their regularities and, perhaps, helping in deciding whether the final object is a fractal.

On the use of wavelets

As we have seen the invariant objects that we want to compute are expressed as graphs of functions (from $\mathbb{S}^1$ to $\mathbb{R}$).

The standard approach to compute invariant objects is to use finite Fourier approximations (trigonometric polynomials) to get expansions as:

$$\mathcal{F}(\theta) = a_0 + \sum_{n=1}^{N} \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right).$$

However, the regularity and periodicity of the trigonometric basis makes clear that this approach is too costly since, as we have seen, the topology and geometry of these objects is extremely complicate.

On the use of wavelets

In this case it seems more natural to use wavelets that adapt much better to oscillatory, irregular and highly discontinuous objects.

Our aim is to devise an algorithm to compute *massive* finite wavelet approximations for attractors with complicate geometry. We need these massive approximations because we want to be able to compute the regularities of the objects.

A (short) crash course on wavelets and regularity

Let us start by the definition of Multiresolution Analysis (MRA)

Definition

A sequence of closed subspaces of $\mathscr{L}^2(\mathbb{R})$, $\{\mathcal{V}_j\}_{j\in\mathbb{Z}}$, is a *Multiresolution Analysis* if it satisfies:

- $\{0\} \subset \cdots \subset \mathcal{V}_1 \subset \mathcal{V}_0 \subset \mathcal{V}_{-1} \subset \cdots \subset \mathscr{L}^2(\mathbb{R})$.
- $\{0\} = \bigcap_{j\in\mathbb{Z}} \mathcal{V}_j$.
- $\text{clos}\left(\bigcup_{j\in\mathbb{Z}} \mathcal{V}_j\right) = \mathscr{L}^2(\mathbb{R})$.
- There exists a function $\phi(x)$ whose *integer translates*, $\phi(x-n)$, form an orthonormal basis of $\mathcal{V}_0$. Such function is called the *scaling function*.
- For each $j \in \mathbb{Z}$ it follows that $f(x) \in \mathcal{V}_j$ if and only if $f(x - 2^j n) \in \mathcal{V}_j$ for each $n \in \mathbb{Z}$.
- For each $j \in \mathbb{Z}$ it follows that $f(x) \in \mathcal{V}_j$ if and only if $f(x/2) \in \mathcal{V}_{j+1}$.

A (short) crash course on wavelets and regularity

Consider the bi-indexed family of maps obtained by dilations and translations of $\phi(x)$:

$$\phi_{j,n}(x) = \frac{1}{\sqrt{2^j}} \phi\left(\frac{x - 2^j n}{2^j}\right).$$

It can be shown that

1. $\{\phi_{j,n}\}_{n\in\mathbb{Z}}$ is an orthonormal basis of $\mathcal{V}_j$ for each $j \in \mathbb{Z}$, and
2. $\phi(x)$ characterizes the whole MRA (see [Mal]).

[Mal] Mallat, Stéphane, *A wavelet tour of signal processing*, Academic Press Inc., San Diego, CA, 1998, xxiv+577.

A (short) crash course on wavelets and regularity

If we fix an MRA, we know that $\mathcal{V}_j \subset \mathcal{V}_{j-1}$. Then we define the subspace *$\mathcal{W}_j$ as the orthogonal complement of $\mathcal{V}_j$ on $\mathcal{V}_{j-1}$*.
That is

$$\mathcal{V}_{j-1} = \mathcal{W}_j \oplus \mathcal{V}_j.$$

We are looking for an orthonormal basis of $\mathcal{W}_j$, *which verifies a relation with $\phi(x)$*, (the *wavelets*). This basis is given from a function called the *mother wavelet* $\psi(x)$ by the formula

$$\psi_{j,n}(x) = \frac{1}{\sqrt{2^j}} \psi\left(\frac{x - 2^j n}{2^j}\right).$$

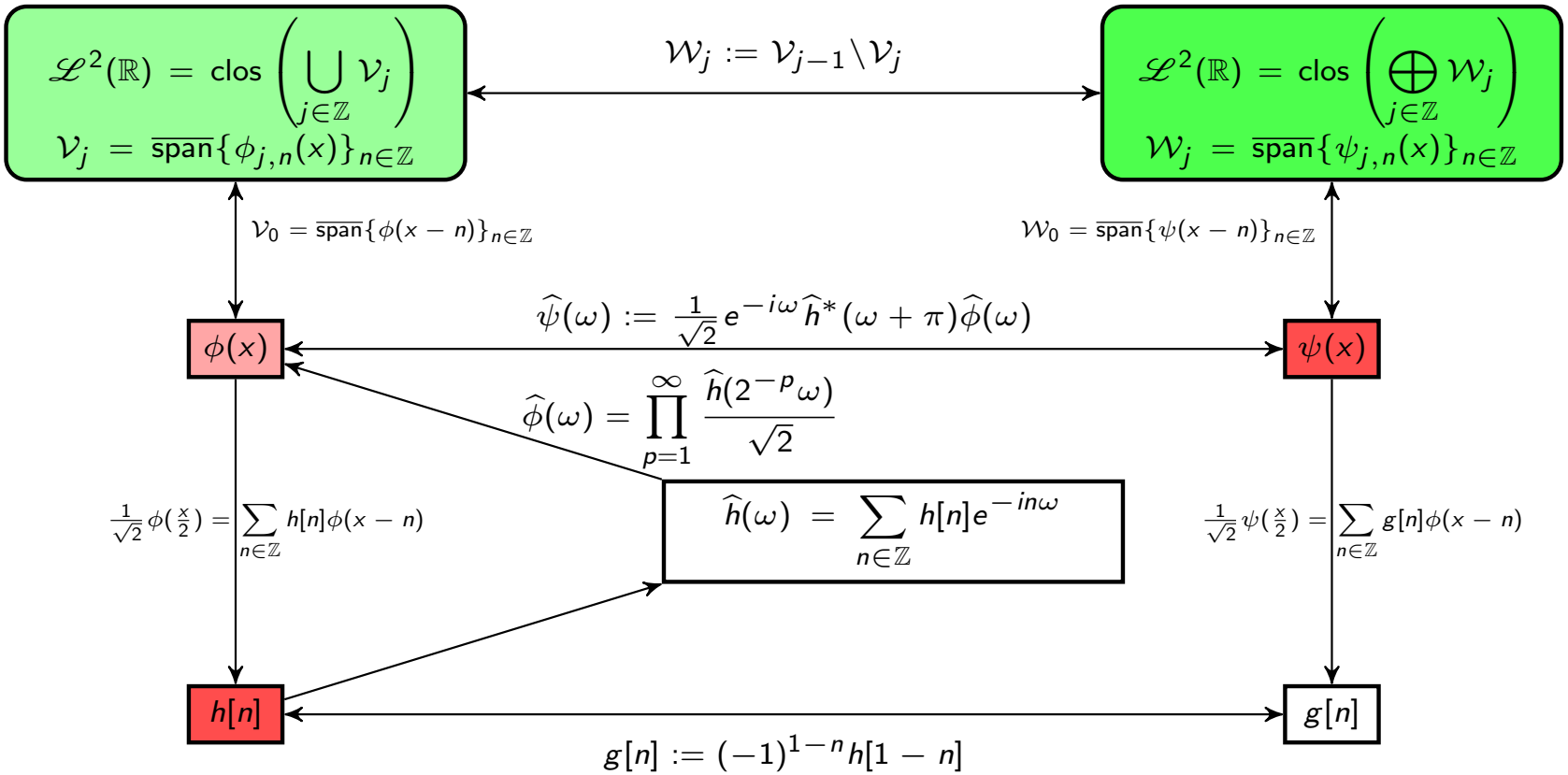
In [Mal] it is shown that:

Mallat and Meyer Theorem

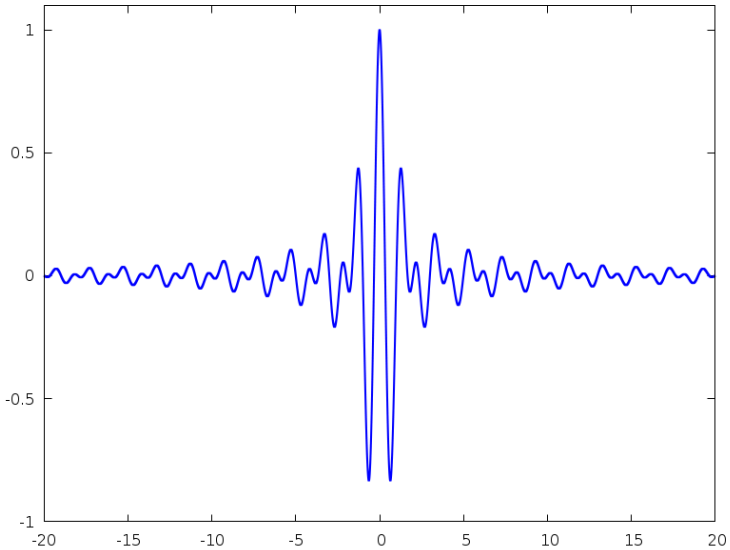
- For every $j \in \mathbb{Z}$ the family $\{\psi_{j,n}\}_{n\in\mathbb{Z}}$ is an orthonormal basis of each $\mathcal{W}_j$,
- The wavelets $\{\psi_{j,n}\}_{(j,n)\in\mathbb{Z}\times\mathbb{Z}}$ are an orthonormal basis of $\mathscr{L}^2(\mathbb{R})$ for all $j, n \in \mathbb{Z}$.

A (short) crash course on wavelets and regularity

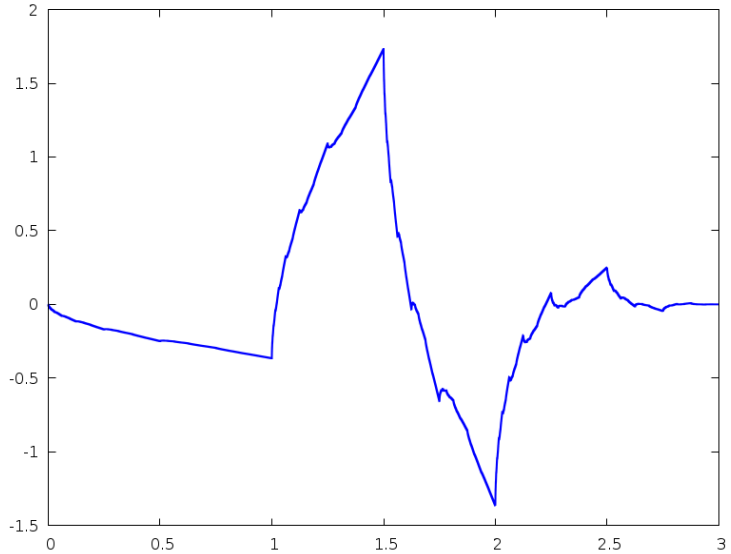
How to compute the mother wavelet



A (short) crash course on wavelets and regularity
Examples of mother wavelets



Shannon wavelet (no compact support)



Daubechies 4 wavelet (compact support)

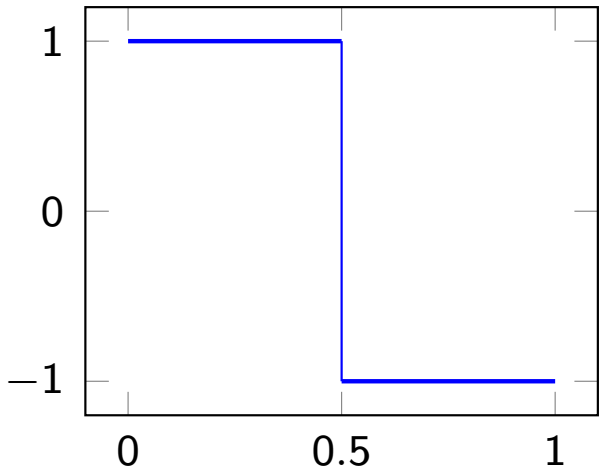
$$\psi(x) = \frac{\sin(2\pi(x-1/2))}{2\pi(x-1/2)} - \frac{\sin(\pi(x-1/2))}{\pi(x-1/2)}$$

$$h[n] = \begin{cases} \frac{\sqrt{2}}{2} & \text{if } n = 0, \\ \sqrt{2}\frac{-1^{(n-1)/2}}{\pi n} & \text{if } n \text{ odd}, \\ 0 & \text{otherwise.} \end{cases}$$

No closed formula

$$h[n] = \begin{cases} 0.4829629131445341\ldots & \text{if } n = 0, \\ 0.8365163037378079\ldots & \text{if } n = 1, \\ 0.2241438680420134\ldots & \text{if } n = 2, \\ -0.1294095225512604\ldots & \text{if } n = 3, \\ 0 & \text{otherwise.} \end{cases}$$

A (short) crash course on wavelets and regularity
Examples of mother wavelets



Haar (Daubechies 2) wavelet

(compact support)

$$\psi(x) := \mathbf{1}_{[0,\frac{1}{2})}(x) - \mathbf{1}_{[\frac{1}{2},1)}(x)$$

where

$$\mathbf{1}_{[a,b)}(x) = \begin{cases} 1 & \text{if } x \in [a,b), \\ 0 & \text{otherwise.} \end{cases}$$

$$h[n] = \begin{cases} \frac{1}{\sqrt{2}} & \text{if } n = 0, 1, \\ 0 & \text{otherwise.} \end{cases}$$

It is the unique Daubechies wavelet with an explicit formula.

Fixing and translating the wavelet

We will be focused on the Daubechies wavelets family. Each Daubechies wavelet has support $[1-p, p]$ where p is the maximal number of vanishing moments:

$$\int_{1-p}^{p} x^k \psi(x)\, dx = 0 \text{ for } 0 \leq k < p.$$

Since our framework is $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$, we transform an $\mathbb{R}$-function into an $\mathbb{S}^1$-function by setting $\psi_{j,n}^{\text{PER}}$ as follows:

$$\psi_{j,n}^{\text{PER}}(\theta) = \sum_{m\in\mathbb{Z}} \psi_{j,n}\overbrace{(\theta+m)}^{x\in\mathbb{R}\ :\ \text{frac}(x)=\theta} = 2^{-j/2} \sum_{m\in\mathbb{Z}} \psi\left(\frac{\overbrace{(\theta+m)}^{x} - 2^j n}{2^j}\right).$$

$\psi_{j,n}^{\text{PER}}$ are 1-periodic functions belonging to $\mathscr{L}^1(\mathbb{S}^1)$.

Fixing and translating the wavelet

It is known that an orthonormal basis of $\mathscr{L}^2(\mathbb{S}^1)$ is given by $\{1, \psi^{\text{PER}}_{-j,n} \text{ with } j \geq 0 \text{ and } n = 0, 1, \ldots, 2^j - 1\}$ provided that $\psi(x)$ is an orthonormal wavelet from a $\mathbb{R}$-MRA (see [HeWe]).

Hence, once ψ is given, we are (almost) ready to compute finite (truncated) wavelet approximations of the type:

$$\varphi \sim a_0 + \sum_{j=0}^{J} \sum_{n=0}^{2^j-1} d_{-j,n} \psi^{\text{PER}}_{-j,n}(\theta).$$

These approximations will be good provided that J is big enough.

Thus, we need to perform a *feasible strategy* to evaluate ψ^{PER} (and $\psi^{\text{PER}}_{-j,n}$) at $\theta \in \mathbb{S}^1$.

[HeWe] Hernández, Eugenio and Weiss, Guido, *A first course on wavelets*, CRC Press, Boca Raton, FL, 1996, xx+489.

Computing regularities with wavelet coefficients

Theorem

Let $s \in \mathbb{R} \setminus \{0\}$ and let ψ be a mother Daubechies wavelet with more than $\max(s, 5/2 - s)$ vanishing moments. Then $f \in \mathscr{B}^s_{\infty,\infty}$ if and only if there exists $C > 0$ such that for all $j \leq 0$

$$\sup_{n \in \mathbb{Z}} |\langle f, \psi^{\mathrm{PER}}_{j,n} \rangle| \leq C 2^{\tau j} \qquad \text{with} \qquad \tau = \begin{cases} s + \frac{1}{2} & \text{if } s > 0, \\ s - \frac{1}{2} & \text{if } s < 0, \end{cases}$$

In the case of Haar, [Tri02], there is an analogous result.


[Coh] Cohen, Albert, *Numerical analysis of wavelet methods*, North-Holland, 2003, xviii+336.


[Tri01] Triebel, Hans, *Theory of function spaces. III*, Birkhäuser Verlag, Basel, 2006, xii+426.


[Tri02] Triebel, Hans, *Bases in function spaces, sampling, discrepancy, numerical integration*, European Mathematical Society, Zürich, 2010, x+296.

Computing regularities with wavelet coefficients

Corollary (Keller's Theorem)

The upper semicontinuous function $\lambda\colon \mathbb{S}^1 \longrightarrow \mathbb{R}^+$ whose graph is the attractor of the system belongs to $\mathscr{B}^s_{\infty,\infty}(\mathbb{S}^1)$ with $s \in (0,1]$ when $\varepsilon > 0$.

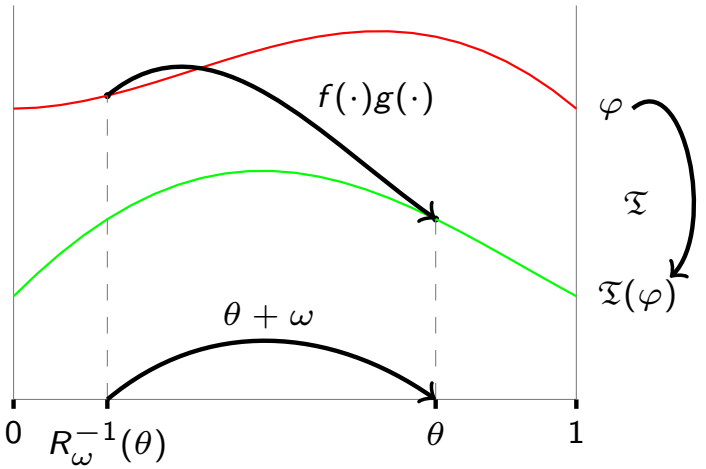
Lemma

The upper semicontinuous function $\lambda\colon \mathbb{S}^1 \longrightarrow \mathbb{R}^+$ whose graph is the attractor of the system belongs to $\mathscr{B}^0_{\infty,\infty}(\mathbb{S}^1)$ when $\varepsilon = 0$.

The above result justifies the use of Besov spaces instead of the Hölder ones because of the *regularity zero*.

Computing coefficients using the Invariance Equation

The functional version of the above systems can be studied using the iteration of the *Transfer Operator*:



Let $\mathscr{P}$ be the space of all functions (not necessarily continuous) from $\mathbb{S}^1$ to $\mathbb{R}$.

Define $\mathfrak{T}(\varphi)(\theta)$ as:

$$\varphi \mapsto f\sigma(\varphi(R_\omega^{-1}(\theta))) \cdot g\varepsilon(R_\omega^{-1}(\theta)).$$

The graph of a function $\varphi\colon \mathbb{S}^1 \longrightarrow \mathbb{R}$ is invariant for the System (2) if and only if φ is a fixed *point* of $\mathfrak{T}$. That is:

$$f\sigma(\varphi(R_\omega^{-1}(\theta))) \cdot g\varepsilon(R_\omega^{-1}(\theta)) = \mathfrak{T}(\varphi)(\theta) = \varphi(\theta).$$

Which is the *Invariance Equation*:

$$f_\sigma(\varphi(\theta)) \cdot g_\varepsilon(\theta) = \varphi(R_\omega(\theta)).$$

Computing coefficients using the Invariance Equation

To solve the above functional equation we write the attractor as

$$\varphi(\theta) = \phi_{0,0} + \sum_{j=0}^{J} \sum_{n=0}^{2^j-1} d_{-j}[n] \psi^{\mathrm{PER}}_{-j,n}(\theta) = d_0 + \sum_{\ell=1}^{N-1} d_\ell \psi^{\mathrm{PER}}_\ell(\theta)$$

where we have set $N = 2^{J+1}$ and, for easiness, we work with the single index $\ell = \ell(j,n) = 2^j + n$ instead of the two indices $-j$ and n.

As usual we plug this expression into the Invariance Equation and we get:

$$d_0 + \sum_{\ell=1}^{N-1} d_\ell \psi^{\mathrm{PER}}_\ell(R_\omega(\theta)) = f_\sigma \left(d_0 + \sum_{\ell=1}^{N-1} d_\ell \psi^{\mathrm{PER}}_\ell(\theta) \right) \cdot g_\varepsilon(\theta)$$

Of course, the coefficients d_0 and d_ℓ are now the explicit numerical unknowns.

Computing coefficients using the Invariance Equation

We have a single functional equation with N unknowns. To compute them, as usual, we discretize the variable θ into N dyadic points

$$\theta_i = \frac{i}{N} \in \mathbb{S}^1 \quad \text{for} \quad i = 0, 1, \ldots, N-1$$

and we impose that the Invariance Equation is verified for every such θ_i:

$$\underbrace{d_0 + \sum_{\ell=1}^{N-1} d_\ell \psi_\ell^{\text{PER}}(R_\omega(\theta_i)) - f_\sigma\left(d_0 + \sum_{\ell=1}^{N-1} d_\ell \psi_\ell^{\text{PER}}(\theta_i)\right) \cdot g_\varepsilon(\theta_i)}_{\mathbf{F}_{\sigma,\varepsilon}(\mathrm{D}^{\text{PER}})_i} = 0$$

where D^{PER} denotes the vector of unknowns:

$$\mathrm{D}^{\text{PER}} := (\phi_{0,0}, d_0[0], \ldots, d_{-J}[2^J - 1]) = (d_0, d_1, \ldots, d_{N-1})$$

Thus, we get a non-linear system of N equations with N unknowns.

Solving $\mathbf{F}_{\sigma,\varepsilon}(\mathrm{D}^{\mathrm{PER}}) = 0$

We will use the Newton's Method to find $\mathrm{D}^{\mathrm{PER}}_{\mathrm{sol}}$ such that

$$\mathbf{F}_{\sigma,\varepsilon}\left(\mathrm{D}^{\mathrm{PER}}_{\mathrm{sol}}\right) \approx \overrightarrow{\mathbf{0}}.$$

That is, given a tolerance `tol`,

$$\begin{cases} \text{starting with a seed } \mathrm{D}^{\mathrm{PER}}_0 \\ \text{solve iteratively } \mathbf{JF}_{\sigma,\varepsilon}(\mathrm{D}^{\mathrm{PER}}_{\mathrm{n}})(\mathrm{X}_{\mathrm{n}}) = -\mathbf{F}_{\sigma,\varepsilon}(\mathrm{D}^{\mathrm{PER}}_{\mathrm{n}}), \\ \qquad\qquad\qquad\qquad \text{where } \mathrm{X}_{\mathrm{n}} := \mathrm{D}^{\mathrm{PER}}_{\mathrm{n}+1} - \mathrm{D}^{\mathrm{PER}}_{\mathrm{n}}, \\ \text{until it is found } \mathrm{D}^{\mathrm{PER}}_{\mathrm{sol}} \text{ such that } |\mathrm{D}^{\mathrm{PER}}_{\mathrm{sol}} - \mathrm{D}^{\mathrm{PER}}_{\mathrm{prev\_sol}}| < \mathtt{tol}. \end{cases}$$

To do this we need to compute the Jacobian matrix

$$\mathbf{JF}_{\sigma,\varepsilon} = \left(\frac{\partial \mathbf{F}_{\sigma,\varepsilon}(\mathrm{D}^{\mathrm{PER}})_i}{\partial d_\ell} \right).$$

Deriving the Jacobian matrix $\mathbf{JF}_{\sigma,\varepsilon}$

Recall that

$$\mathbf{F}_{\sigma,\varepsilon}(\mathrm{D}^{\mathrm{PER}})_i = d_0 + \sum_{\ell=1}^{N-1} d_\ell \psi_\ell^{\mathrm{PER}}(R_\omega(\theta_i)) - f_\sigma\left(d_0 + \sum_{\ell=1}^{N-1} d_\ell \psi_\ell^{\mathrm{PER}}(\theta_i)\right) \cdot g_\varepsilon(\theta_i).$$

Then, the entry i, ℓ of the Jacobian matrix which is

$$(\mathbf{JF}_{\sigma,\varepsilon})_{i,\ell} = \frac{\partial \mathbf{F}_{\sigma,\varepsilon}(\mathrm{D}^{\mathrm{PER}})_i}{\partial d_\ell}$$

can be explicitly computed as:

$$(\mathbf{JF}_{\sigma,\varepsilon})_{i,\ell} = \begin{cases} 1 - f'_\sigma\left(d_0 + \sum_{\ell=1}^{N-1} d_\ell \psi_\ell^{\mathrm{PER}}(\theta_i)\right) \cdot g_\varepsilon(\theta_i) & \text{if } \ell = 0, \\ \psi_\ell^{\mathrm{PER}}(R_\omega(\theta_i)) - f'_\sigma\left(d_0 + \sum_{\ell=1}^{N-1} d_\ell \psi_\ell^{\mathrm{PER}}(\theta_i)\right) \cdot g_\varepsilon(\theta_i) \cdot \psi_\ell^{\mathrm{PER}}(\theta_i) & \text{if } \ell \neq 0. \end{cases}$$

These equations in explicit form (and also the invariance equation) are too complicate to deal with. It is much easier to use the algebraized version of these equations.

Going into algebra: The wavelets and rotated wavelets matrices Ψ and Ψ_R

We start by defining the following $N \times N$ matrices:

The wavelets and rotated wavelets matrices

- *The wavelets matrix* Ψ whose column 0 is constant 1 and the column $\ell \in \{1, 2, \ldots, N-1\}$ is $\psi_\ell^{\text{PER}}(\theta_i)$ with $i = 0, 1, \ldots, N-1$,
- *The rotated wavelets matrix* Ψ_R whose column 0 is constant 1 and the column $\ell \in \{1, 2, \ldots, N-1\}$ is $\psi_\ell^{\text{PER}}(R_\omega(\theta_i))$ with $i = 0, 1, \ldots, N-1$.

Structure of the matrices Ψ and Ψ_R

$$
\begin{pmatrix}
1 & \psi^{\mathrm{PER}}_{0,0}(0) & \psi^{\mathrm{PER}}_{-1,0}(0) & \psi^{\mathrm{PER}}_{-1,1}(0) & \psi^{\mathrm{PER}}_{-2,0}(0) & \psi^{\mathrm{PER}}_{-2,1}(0) & \psi^{\mathrm{PER}}_{-2,2}(0) & \psi^{\mathrm{PER}}_{-2,3}(0) & \psi^{\mathrm{PER}}_{-3,0}(0) & \psi^{\mathrm{PER}}_{-3,1}(0) & \psi^{\mathrm{PER}}_{-3,2}(0) & \psi^{\mathrm{PER}}_{-3,3}(0) & \psi^{\mathrm{PER}}_{-3,4}(0) & \psi^{\mathrm{PER}}_{-3,5}(0) & \psi^{\mathrm{PER}}_{-3,6}(0) & \psi^{\mathrm{PER}}_{-3,7}(0) \\
1 & \psi^{\mathrm{PER}}_{0,0}\left(\frac{1}{16}\right) & \psi^{\mathrm{PER}}_{-1,0}\left(\frac{1}{16}\right) & \psi^{\mathrm{PER}}_{-1,1}\left(\frac{1}{16}\right) & \psi^{\mathrm{PER}}_{-2,0}\left(\frac{1}{16}\right) & \psi^{\mathrm{PER}}_{-2,1}\left(\frac{1}{16}\right) & \psi^{\mathrm{PER}}_{-2,2}\left(\frac{1}{16}\right) & \psi^{\mathrm{PER}}_{-2,3}\left(\frac{1}{16}\right) & \psi^{\mathrm{PER}}_{-3,0}\left(\frac{1}{16}\right) & \psi^{\mathrm{PER}}_{-3,1}\left(\frac{1}{16}\right) & \psi^{\mathrm{PER}}_{-3,2}\left(\frac{1}{16}\right) & \psi^{\mathrm{PER}}_{-3,3}\left(\frac{1}{16}\right) & \psi^{\mathrm{PER}}_{-3,4}\left(\frac{1}{16}\right) & \psi^{\mathrm{PER}}_{-3,5}\left(\frac{1}{16}\right) & \psi^{\mathrm{PER}}_{-3,6}\left(\frac{1}{16}\right) & \psi^{\mathrm{PER}}_{-3,7}\left(\frac{1}{16}\right) \\
1 & \psi^{\mathrm{PER}}_{0,0}\left(\frac{2}{16}\right) & \psi^{\mathrm{PER}}_{-1,0}\left(\frac{2}{16}\right) & \psi^{\mathrm{PER}}_{-1,1}\left(\frac{2}{16}\right) & \psi^{\mathrm{PER}}_{-2,0}\left(\frac{2}{16}\right) & \psi^{\mathrm{PER}}_{-2,1}\left(\frac{2}{16}\right) & \psi^{\mathrm{PER}}_{-2,2}\left(\frac{2}{16}\right) & \psi^{\mathrm{PER}}_{-2,3}\left(\frac{2}{16}\right) & \psi^{\mathrm{PER}}_{-3,0}\left(\frac{2}{16}\right) & \psi^{\mathrm{PER}}_{-3,1}\left(\frac{2}{16}\right) & \psi^{\mathrm{PER}}_{-3,2}\left(\frac{2}{16}\right) & \psi^{\mathrm{PER}}_{-3,3}\left(\frac{2}{16}\right) & \psi^{\mathrm{PER}}_{-3,4}\left(\frac{2}{16}\right) & \psi^{\mathrm{PER}}_{-3,5}\left(\frac{2}{16}\right) & \psi^{\mathrm{PER}}_{-3,6}\left(\frac{2}{16}\right) & \psi^{\mathrm{PER}}_{-3,7}\left(\frac{2}{16}\right) \\
1 & \psi^{\mathrm{PER}}_{0,0}\left(\frac{3}{16}\right) & \psi^{\mathrm{PER}}_{-1,0}\left(\frac{3}{16}\right) & \psi^{\mathrm{PER}}_{-1,1}\left(\frac{3}{16}\right) & \psi^{\mathrm{PER}}_{-2,0}\left(\frac{3}{16}\right) & \psi^{\mathrm{PER}}_{-2,1}\left(\frac{3}{16}\right) & \psi^{\mathrm{PER}}_{-2,2}\left(\frac{3}{16}\right) & \psi^{\mathrm{PER}}_{-2,3}\left(\frac{3}{16}\right) & \psi^{\mathrm{PER}}_{-3,0}\left(\frac{3}{16}\right) & \psi^{\mathrm{PER}}_{-3,1}\left(\frac{3}{16}\right) & \psi^{\mathrm{PER}}_{-3,2}\left(\frac{3}{16}\right) & \psi^{\mathrm{PER}}_{-3,3}\left(\frac{3}{16}\right) & \psi^{\mathrm{PER}}_{-3,4}\left(\frac{3}{16}\right) & \psi^{\mathrm{PER}}_{-3,5}\left(\frac{3}{16}\right) & \psi^{\mathrm{PER}}_{-3,6}\left(\frac{3}{16}\right) & \psi^{\mathrm{PER}}_{-3,7}\left(\frac{3}{16}\right) \\
1 & \psi^{\mathrm{PER}}_{0,0}\left(\frac{4}{16}\right) & \psi^{\mathrm{PER}}_{-1,0}\left(\frac{4}{16}\right) & \psi^{\mathrm{PER}}_{-1,1}\left(\frac{4}{16}\right) & \psi^{\mathrm{PER}}_{-2,0}\left(\frac{4}{16}\right) & \psi^{\mathrm{PER}}_{-2,1}\left(\frac{4}{16}\right) & \psi^{\mathrm{PER}}_{-2,2}\left(\frac{4}{16}\right) & \psi^{\mathrm{PER}}_{-2,3}\left(\frac{4}{16}\right) & \psi^{\mathrm{PER}}_{-3,0}\left(\frac{4}{16}\right) & \psi^{\mathrm{PER}}_{-3,1}\left(\frac{4}{16}\right) & \psi^{\mathrm{PER}}_{-3,2}\left(\frac{4}{16}\right) & \psi^{\mathrm{PER}}_{-3,3}\left(\frac{4}{16}\right) & \psi^{\mathrm{PER}}_{-3,4}\left(\frac{4}{16}\right) & \psi^{\mathrm{PER}}_{-3,5}\left(\frac{4}{16}\right) & \psi^{\mathrm{PER}}_{-3,6}\left(\frac{4}{16}\right) & \psi^{\mathrm{PER}}_{-3,7}\left(\frac{4}{16}\right) \\
1 & \psi^{\mathrm{PER}}_{0,0}\left(\frac{5}{16}\right) & \psi^{\mathrm{PER}}_{-1,0}\left(\frac{5}{16}\right) & \psi^{\mathrm{PER}}_{-1,1}\left(\frac{5}{16}\right) & \psi^{\mathrm{PER}}_{-2,0}\left(\frac{5}{16}\right) & \psi^{\mathrm{PER}}_{-2,1}\left(\frac{5}{16}\right) & \psi^{\mathrm{PER}}_{-2,2}\left(\frac{5}{16}\right) & \psi^{\mathrm{PER}}_{-2,3}\left(\frac{5}{16}\right) & \psi^{\mathrm{PER}}_{-3,0}\left(\frac{5}{16}\right) & \psi^{\mathrm{PER}}_{-3,1}\left(\frac{5}{16}\right) & \psi^{\mathrm{PER}}_{-3,2}\left(\frac{5}{16}\right) & \psi^{\mathrm{PER}}_{-3,3}\left(\frac{5}{16}\right) & \psi^{\mathrm{PER}}_{-3,4}\left(\frac{5}{16}\right) & \psi^{\mathrm{PER}}_{-3,5}\left(\frac{5}{16}\right) & \psi^{\mathrm{PER}}_{-3,6}\left(\frac{5}{16}\right) & \psi^{\mathrm{PER}}_{-3,7}\left(\frac{5}{16}\right) \\
1 & \psi^{\mathrm{PER}}_{0,0}\left(\frac{6}{16}\right) & \psi^{\mathrm{PER}}_{-1,0}\left(\frac{6}{16}\right) & \psi^{\mathrm{PER}}_{-1,1}\left(\frac{6}{16}\right) & \psi^{\mathrm{PER}}_{-2,0}\left(\frac{6}{16}\right) & \psi^{\mathrm{PER}}_{-2,1}\left(\frac{6}{16}\right) & \psi^{\mathrm{PER}}_{-2,2}\left(\frac{6}{16}\right) & \psi^{\mathrm{PER}}_{-2,3}\left(\frac{6}{16}\right) & \psi^{\mathrm{PER}}_{-3,0}\left(\frac{6}{16}\right) & \psi^{\mathrm{PER}}_{-3,1}\left(\frac{6}{16}\right) & \psi^{\mathrm{PER}}_{-3,2}\left(\frac{6}{16}\right) & \psi^{\mathrm{PER}}_{-3,3}\left(\frac{6}{16}\right) & \psi^{\mathrm{PER}}_{-3,4}\left(\frac{6}{16}\right) & \psi^{\mathrm{PER}}_{-3,5}\left(\frac{6}{16}\right) & \psi^{\mathrm{PER}}_{-3,6}\left(\frac{6}{16}\right) & \psi^{\mathrm{PER}}_{-3,7}\left(\frac{6}{16}\right) \\
1 & \psi^{\mathrm{PER}}_{0,0}\left(\frac{7}{16}\right) & \psi^{\mathrm{PER}}_{-1,0}\left(\frac{7}{16}\right) & \psi^{\mathrm{PER}}_{-1,1}\left(\frac{7}{16}\right) & \psi^{\mathrm{PER}}_{-2,0}\left(\frac{7}{16}\right) & \psi^{\mathrm{PER}}_{-2,1}\left(\frac{7}{16}\right) & \psi^{\mathrm{PER}}_{-2,2}\left(\frac{7}{16}\right) & \psi^{\mathrm{PER}}_{-2,3}\left(\frac{7}{16}\right) & \psi^{\mathrm{PER}}_{-3,0}\left(\frac{7}{16}\right) & \psi^{\mathrm{PER}}_{-3,1}\left(\frac{7}{16}\right) & \psi^{\mathrm{PER}}_{-3,2}\left(\frac{7}{16}\right) & \psi^{\mathrm{PER}}_{-3,3}\left(\frac{7}{16}\right) & \psi^{\mathrm{PER}}_{-3,4}\left(\frac{7}{16}\right) & \psi^{\mathrm{PER}}_{-3,5}\left(\frac{7}{16}\right) & \psi^{\mathrm{PER}}_{-3,6}\left(\frac{7}{16}\right) & \psi^{\mathrm{PER}}_{-3,7}\left(\frac{7}{16}\right) \\
1 & \psi^{\mathrm{PER}}_{0,0}\left(\frac{8}{16}\right) & \psi^{\mathrm{PER}}_{-1,0}\left(\frac{8}{16}\right) & \psi^{\mathrm{PER}}_{-1,1}\left(\frac{8}{16}\right) & \psi^{\mathrm{PER}}_{-2,0}\left(\frac{8}{16}\right) & \psi^{\mathrm{PER}}_{-2,1}\left(\frac{8}{16}\right) & \psi^{\mathrm{PER}}_{-2,2}\left(\frac{8}{16}\right) & \psi^{\mathrm{PER}}_{-2,3}\left(\frac{8}{16}\right) & \psi^{\mathrm{PER}}_{-3,0}\left(\frac{8}{16}\right) & \psi^{\mathrm{PER}}_{-3,1}\left(\frac{8}{16}\right) & \psi^{\mathrm{PER}}_{-3,2}\left(\frac{8}{16}\right) & \psi^{\mathrm{PER}}_{-3,3}\left(\frac{8}{16}\right) & \psi^{\mathrm{PER}}_{-3,4}\left(\frac{8}{16}\right) & \psi^{\mathrm{PER}}_{-3,5}\left(\frac{8}{16}\right) & \psi^{\mathrm{PER}}_{-3,6}\left(\frac{8}{16}\right) & \psi^{\mathrm{PER}}_{-3,7}\left(\frac{8}{16}\right) \\
1 & \psi^{\mathrm{PER}}_{0,0}\left(\frac{9}{16}\right) & \psi^{\mathrm{PER}}_{-1,0}\left(\frac{9}{16}\right) & \psi^{\mathrm{PER}}_{-1,1}\left(\frac{9}{16}\right) & \psi^{\mathrm{PER}}_{-2,0}\left(\frac{9}{16}\right) & \psi^{\mathrm{PER}}_{-2,1}\left(\frac{9}{16}\right) & \psi^{\mathrm{PER}}_{-2,2}\left(\frac{9}{16}\right) & \psi^{\mathrm{PER}}_{-2,3}\left(\frac{9}{16}\right) & \psi^{\mathrm{PER}}_{-3,0}\left(\frac{9}{16}\right) & \psi^{\mathrm{PER}}_{-3,1}\left(\frac{9}{16}\right) & \psi^{\mathrm{PER}}_{-3,2}\left(\frac{9}{16}\right) & \psi^{\mathrm{PER}}_{-3,3}\left(\frac{9}{16}\right) & \psi^{\mathrm{PER}}_{-3,4}\left(\frac{9}{16}\right) & \psi^{\mathrm{PER}}_{-3,5}\left(\frac{9}{16}\right) & \psi^{\mathrm{PER}}_{-3,6}\left(\frac{9}{16}\right) & \psi^{\mathrm{PER}}_{-3,7}\left(\frac{9}{16}\right) \\
1 & \psi^{\mathrm{PER}}_{0,0}\left(\frac{10}{16}\right) & \psi^{\mathrm{PER}}_{-1,0}\left(\frac{10}{16}\right) & \psi^{\mathrm{PER}}_{-1,1}\left(\frac{10}{16}\right) & \psi^{\mathrm{PER}}_{-2,0}\left(\frac{10}{16}\right) & \psi^{\mathrm{PER}}_{-2,1}\left(\frac{10}{16}\right) & \psi^{\mathrm{PER}}_{-2,2}\left(\frac{10}{16}\right) & \psi^{\mathrm{PER}}_{-2,3}\left(\frac{10}{16}\right) & \psi^{\mathrm{PER}}_{-3,0}\left(\frac{10}{16}\right) & \psi^{\mathrm{PER}}_{-3,1}\left(\frac{10}{16}\right) & \psi^{\mathrm{PER}}_{-3,2}\left(\frac{10}{16}\right) & \psi^{\mathrm{PER}}_{-3,3}\left(\frac{10}{16}\right) & \psi^{\mathrm{PER}}_{-3,4}\left(\frac{10}{16}\right) & \psi^{\mathrm{PER}}_{-3,5}\left(\frac{10}{16}\right) & \psi^{\mathrm{PER}}_{-3,6}\left(\frac{10}{16}\right) & \psi^{\mathrm{PER}}_{-3,7}\left(\frac{10}{16}\right) \\
1 & \psi^{\mathrm{PER}}_{0,0}\left(\frac{11}{16}\right) & \psi^{\mathrm{PER}}_{-1,0}\left(\frac{11}{16}\right) & \psi^{\mathrm{PER}}_{-1,1}\left(\frac{11}{16}\right) & \psi^{\mathrm{PER}}_{-2,0}\left(\frac{11}{16}\right) & \psi^{\mathrm{PER}}_{-2,1}\left(\frac{11}{16}\right) & \psi^{\mathrm{PER}}_{-2,2}\left(\frac{11}{16}\right) & \psi^{\mathrm{PER}}_{-2,3}\left(\frac{11}{16}\right) & \psi^{\mathrm{PER}}_{-3,0}\left(\frac{11}{16}\right) & \psi^{\mathrm{PER}}_{-3,1}\left(\frac{11}{16}\right) & \psi^{\mathrm{PER}}_{-3,2}\left(\frac{11}{16}\right) & \psi^{\mathrm{PER}}_{-3,3}\left(\frac{11}{16}\right) & \psi^{\mathrm{PER}}_{-3,4}\left(\frac{11}{16}\right) & \psi^{\mathrm{PER}}_{-3,5}\left(\frac{11}{16}\right) & \psi^{\mathrm{PER}}_{-3,6}\left(\frac{11}{16}\right) & \psi^{\mathrm{PER}}_{-3,7}\left(\frac{11}{16}\right) \\
1 & \psi^{\mathrm{PER}}_{0,0}\left(\frac{12}{16}\right) & \psi^{\mathrm{PER}}_{-1,0}\left(\frac{12}{16}\right) & \psi^{\mathrm{PER}}_{-1,1}\left(\frac{12}{16}\right) & \psi^{\mathrm{PER}}_{-2,0}\left(\frac{12}{16}\right) & \psi^{\mathrm{PER}}_{-2,1}\left(\frac{12}{16}\right) & \psi^{\mathrm{PER}}_{-2,2}\left(\frac{12}{16}\right) & \psi^{\mathrm{PER}}_{-2,3}\left(\frac{12}{16}\right) & \psi^{\mathrm{PER}}_{-3,0}\left(\frac{12}{16}\right) & \psi^{\mathrm{PER}}_{-3,1}\left(\frac{12}{16}\right) & \psi^{\mathrm{PER}}_{-3,2}\left(\frac{12}{16}\right) & \psi^{\mathrm{PER}}_{-3,3}\left(\frac{12}{16}\right) & \psi^{\mathrm{PER}}_{-3,4}\left(\frac{12}{16}\right) & \psi^{\mathrm{PER}}_{-3,5}\left(\frac{12}{16}\right) & \psi^{\mathrm{PER}}_{-3,6}\left(\frac{12}{16}\right) & \psi^{\mathrm{PER}}_{-3,7}\left(\frac{12}{16}\right) \\
1 & \psi^{\mathrm{PER}}_{0,0}\left(\frac{13}{16}\right) & \psi^{\mathrm{PER}}_{-1,0}\left(\frac{13}{16}\right) & \psi^{\mathrm{PER}}_{-1,1}\left(\frac{13}{16}\right) & \psi^{\mathrm{PER}}_{-2,0}\left(\frac{13}{16}\right) & \psi^{\mathrm{PER}}_{-2,1}\left(\frac{13}{16}\right) & \psi^{\mathrm{PER}}_{-2,2}\left(\frac{13}{16}\right) & \psi^{\mathrm{PER}}_{-2,3}\left(\frac{13}{16}\right) & \psi^{\mathrm{PER}}_{-3,0}\left(\frac{13}{16}\right) & \psi^{\mathrm{PER}}_{-3,1}\left(\frac{13}{16}\right) & \psi^{\mathrm{PER}}_{-3,2}\left(\frac{13}{16}\right) & \psi^{\mathrm{PER}}_{-3,3}\left(\frac{13}{16}\right) & \psi^{\mathrm{PER}}_{-3,4}\left(\frac{13}{16}\right) & \psi^{\mathrm{PER}}_{-3,5}\left(\frac{13}{16}\right) & \psi^{\mathrm{PER}}_{-3,6}\left(\frac{13}{16}\right) & \psi^{\mathrm{PER}}_{-3,7}\left(\frac{13}{16}\right) \\
1 & \psi^{\mathrm{PER}}_{0,0}\left(\frac{14}{16}\right) & \psi^{\mathrm{PER}}_{-1,0}\left(\frac{14}{16}\right) & \psi^{\mathrm{PER}}_{-1,1}\left(\frac{14}{16}\right) & \psi^{\mathrm{PER}}_{-2,0}\left(\frac{14}{16}\right) & \psi^{\mathrm{PER}}_{-2,1}\left(\frac{14}{16}\right) & \psi^{\mathrm{PER}}_{-2,2}\left(\frac{14}{16}\right) & \psi^{\mathrm{PER}}_{-2,3}\left(\frac{14}{16}\right) & \psi^{\mathrm{PER}}_{-3,0}\left(\frac{14}{16}\right) & \psi^{\mathrm{PER}}_{-3,1}\left(\frac{14}{16}\right) & \psi^{\mathrm{PER}}_{-3,2}\left(\frac{14}{16}\right) & \psi^{\mathrm{PER}}_{-3,3}\left(\frac{14}{16}\right) & \psi^{\mathrm{PER}}_{-3,4}\left(\frac{14}{16}\right) & \psi^{\mathrm{PER}}_{-3,5}\left(\frac{14}{16}\right) & \psi^{\mathrm{PER}}_{-3,6}\left(\frac{14}{16}\right) & \psi^{\mathrm{PER}}_{-3,7}\left(\frac{14}{16}\right) \\
1 & \psi^{\mathrm{PER}}_{0,0}\left(\frac{15}{16}\right) & \psi^{\mathrm{PER}}_{-1,0}\left(\frac{15}{16}\right) & \psi^{\mathrm{PER}}_{-1,1}\left(\frac{15}{16}\right) & \psi^{\mathrm{PER}}_{-2,0}\left(\frac{15}{16}\right) & \psi^{\mathrm{PER}}_{-2,1}\left(\frac{15}{16}\right) & \psi^{\mathrm{PER}}_{-2,2}\left(\frac{15}{16}\right) & \psi^{\mathrm{PER}}_{-2,3}\left(\frac{15}{16}\right) & \psi^{\mathrm{PER}}_{-3,0}\left(\frac{15}{16}\right) & \psi^{\mathrm{PER}}_{-3,1}\left(\frac{15}{16}\right) & \psi^{\mathrm{PER}}_{-3,2}\left(\frac{15}{16}\right) & \psi^{\mathrm{PER}}_{-3,3}\left(\frac{15}{16}\right) & \psi^{\mathrm{PER}}_{-3,4}\left(\frac{15}{16}\right) & \psi^{\mathrm{PER}}_{-3,5}\left(\frac{15}{16}\right) & \psi^{\mathrm{PER}}_{-3,6}\left(\frac{15}{16}\right) & \psi^{\mathrm{PER}}_{-3,7}\left(\frac{15}{16}\right)
\end{pmatrix}
$$

The rows of Ψ_R are evaluated at $R_\omega(\theta_i) = \left(\frac{i}{N} + \omega\right) \pmod 1$ instead of at $\theta_i = \frac{i}{N}$.

Going into algebra: The reconstruction of the attractor from the wavelet coefficients

With the above notation, we clearly get

$$\varphi(\theta_i) = d_0 + \sum_{\ell=1}^{N-1} d_\ell \psi_\ell^{\mathrm{PER}}(\theta_i) = [\Psi \mathrm{D}^{\mathrm{PER}}]_i.$$

Hence,

THE RECONSTRUCTION

$$(\varphi(\theta_0), \varphi(\theta_1), \ldots, \varphi(\theta_{N-1})^\top = \Psi \mathrm{D}^{\mathrm{PER}}.$$

Going into algebra: Easy to rewrite

Algebraic expression of $\mathbf{F}_{\sigma,\varepsilon}(\mathrm{D}_n^{\mathrm{PER}})$

$$\mathbf{F}_{\sigma,\varepsilon}(\mathrm{D}_n^{\mathrm{PER}}) = \Psi_R \mathrm{D}_n^{\mathrm{PER}} - \wp_n,$$

where $\wp_n$ is the N-dimensional vector that has as $i-$th component:

$$[\wp_n]_i = f_\sigma\left([\Psi \mathrm{D}_n^{\mathrm{PER}}]_i\right) \cdot g_\varepsilon(\theta_i).$$

Algebraic form of $\mathbf{JF}_{\sigma,\varepsilon}$

$$\mathbf{JF}_{\sigma,\varepsilon} = \Psi_R - \Delta_{\sigma,\varepsilon}\Psi$$

where $\Delta_{\sigma,\varepsilon}$ is the $N \times N$ diagonal matrix whose (diagonal) entries are:

$$\frac{\partial F_{\sigma,\varepsilon}}{\partial x} = f'_\sigma([\Psi \mathrm{D}^{\mathrm{PER}}]_i) \cdot g_\varepsilon(\theta_i) = \frac{\partial\,[\wp_n]_i}{\partial x}.$$

Remark

The above algebraic form implies that the matrices Ψ and Ψ_R only need to be (pre)computed once in the whole computation.

The equations in algebraic form

At each Newton iterate we have to solve

$$\wp_n - \Psi_R D_n^{PER} = -\mathbf{F}_{\sigma,\varepsilon}(D_n^{PER}) = \mathbf{JF}_{\sigma,\varepsilon}(D_n^{PER})(X_n) = (\Psi_R - \Delta_{\sigma,\varepsilon}\Psi)X_n$$

A remark on the initialization: how to choose the seed

Using the Trapezoidal rule we have

$$d_\ell := \int_{\mathbb{S}^1} \psi_\ell^{PER}\varphi \, d\theta \approx \frac{1}{N}\sum_{i=0}^{N-1} \psi_\ell^{PER}(\theta_i)\varphi(\theta_i)$$

$$= \frac{1}{N}\left[\Psi^\top \left(\varphi(\theta_0), \varphi(\theta_1), \ldots, \varphi(\theta_{N-1})\right)^\top\right]_\ell .$$

Then, we obtain

A good seed:

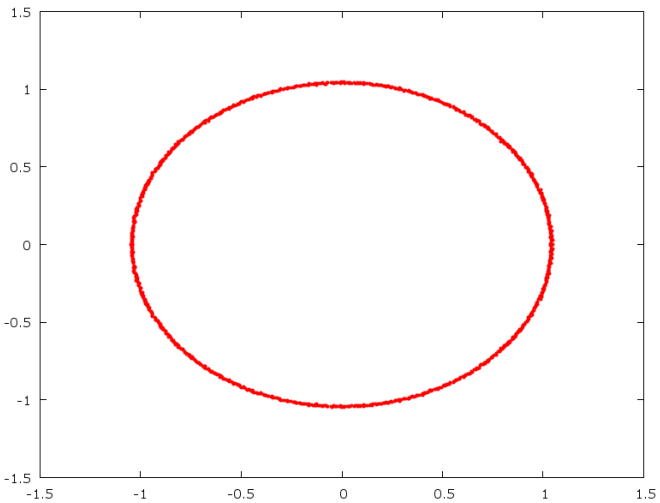
$$D_0^{PER} := \frac{1}{N}\Psi^\top \left(\varphi(\theta_0), \varphi(\theta_1), \ldots, \varphi(\theta_{N-1})\right)^\top$$

The seed and the linear system from Newton's method

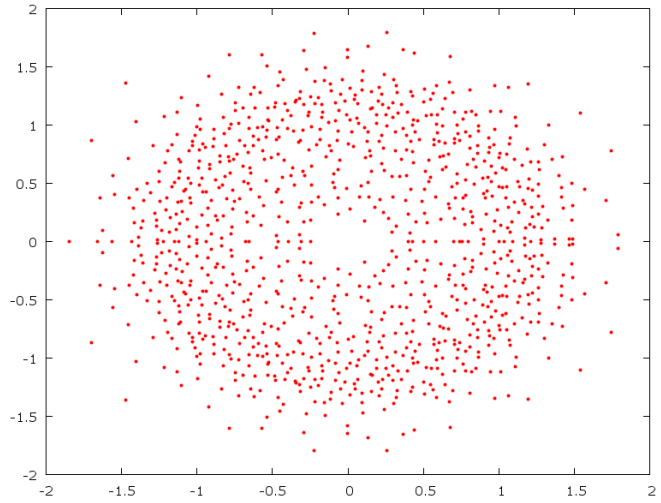
We have to solve many times the system

$$(\Psi_R - \Delta_{\sigma,\varepsilon}\Psi)\mathsf{X}_\mathsf{n} = -\mathbf{F}_{\sigma,\varepsilon}(\mathsf{D}_\mathsf{n}^{\mathsf{PER}})$$

This is difficult to solve because we are interested in doing so for N big ($2^{2?}$) and, moreover, its typical spectrum looks like:



Eigenvalues for a non-pinched case.



Eigenvalues for a quasi-pinched case.

A solution to both problems is *preconditioning*.

Using Haar to solve the Invariance Equation: preconditioning

A *right preconditioning* of a system $Ax = b$

is to *solve first* $(A\mathbf{R}_{\text{prec}})y = b$ and, *after*, calculate $\mathbf{R}_{\text{prec}}^{-1}x = y$ to get the true solution x, where $\mathbf{R}_{\text{prec}}$ is an *appropriate* (easy) non-singular (easily invertible) matrix.

To see how to choose a good candidate for $\mathbf{R}_{\text{prec}}$ let us study better the Haar wavelet matrices.

When the matrix Ψ generates Ψ_R: The trick for Haar

The Haar matrix Ψ for $J = 2$ ($N = 2^{J+1} = 8$) is:

$$\Psi = \begin{pmatrix} 1 & \psi^{\mathrm{PER}}_{0,0}(0) & \psi^{\mathrm{PER}}_{-1,0}(0) & \psi^{\mathrm{PER}}_{-1,1}(0) & \psi^{\mathrm{PER}}_{-2,0}(0) & \psi^{\mathrm{PER}}_{-2,1}(0) & \psi^{\mathrm{PER}}_{-2,2}(0) & \psi^{\mathrm{PER}}_{-2,3}(0) \\ 1 & \psi^{\mathrm{PER}}_{0,0}\left(\frac{1}{8}\right) & \psi^{\mathrm{PER}}_{-1,0}\left(\frac{1}{8}\right) & \psi^{\mathrm{PER}}_{-1,1}\left(\frac{1}{8}\right) & \psi^{\mathrm{PER}}_{-2,0}\left(\frac{1}{8}\right) & \psi^{\mathrm{PER}}_{-2,1}\left(\frac{1}{8}\right) & \psi^{\mathrm{PER}}_{-2,2}\left(\frac{1}{8}\right) & \psi^{\mathrm{PER}}_{-2,3}\left(\frac{1}{8}\right) \\ 1 & \psi^{\mathrm{PER}}_{0,0}\left(\frac{2}{8}\right) & \psi^{\mathrm{PER}}_{-1,0}\left(\frac{2}{8}\right) & \psi^{\mathrm{PER}}_{-1,1}\left(\frac{2}{8}\right) & \psi^{\mathrm{PER}}_{-2,0}\left(\frac{2}{8}\right) & \psi^{\mathrm{PER}}_{-2,1}\left(\frac{2}{8}\right) & \psi^{\mathrm{PER}}_{-2,2}\left(\frac{2}{8}\right) & \psi^{\mathrm{PER}}_{-2,3}\left(\frac{2}{8}\right) \\ 1 & \psi^{\mathrm{PER}}_{0,0}\left(\frac{3}{8}\right) & \psi^{\mathrm{PER}}_{-1,0}\left(\frac{3}{8}\right) & \psi^{\mathrm{PER}}_{-1,1}\left(\frac{3}{8}\right) & \psi^{\mathrm{PER}}_{-2,0}\left(\frac{3}{8}\right) & \psi^{\mathrm{PER}}_{-2,1}\left(\frac{3}{8}\right) & \psi^{\mathrm{PER}}_{-2,2}\left(\frac{3}{8}\right) & \psi^{\mathrm{PER}}_{-2,3}\left(\frac{3}{8}\right) \\ 1 & \psi^{\mathrm{PER}}_{0,0}\left(\frac{4}{8}\right) & \psi^{\mathrm{PER}}_{-1,0}\left(\frac{4}{8}\right) & \psi^{\mathrm{PER}}_{-1,1}\left(\frac{4}{8}\right) & \psi^{\mathrm{PER}}_{-2,0}\left(\frac{4}{8}\right) & \psi^{\mathrm{PER}}_{-2,1}\left(\frac{4}{8}\right) & \psi^{\mathrm{PER}}_{-2,2}\left(\frac{4}{8}\right) & \psi^{\mathrm{PER}}_{-2,3}\left(\frac{4}{8}\right) \\ 1 & \psi^{\mathrm{PER}}_{0,0}\left(\frac{5}{8}\right) & \psi^{\mathrm{PER}}_{-1,0}\left(\frac{5}{8}\right) & \psi^{\mathrm{PER}}_{-1,1}\left(\frac{5}{8}\right) & \psi^{\mathrm{PER}}_{-2,0}\left(\frac{5}{8}\right) & \psi^{\mathrm{PER}}_{-2,1}\left(\frac{5}{8}\right) & \psi^{\mathrm{PER}}_{-2,2}\left(\frac{5}{8}\right) & \psi^{\mathrm{PER}}_{-2,3}\left(\frac{5}{8}\right) \\ 1 & \psi^{\mathrm{PER}}_{0,0}\left(\frac{6}{8}\right) & \psi^{\mathrm{PER}}_{-1,0}\left(\frac{6}{8}\right) & \psi^{\mathrm{PER}}_{-1,1}\left(\frac{6}{8}\right) & \psi^{\mathrm{PER}}_{-2,0}\left(\frac{6}{8}\right) & \psi^{\mathrm{PER}}_{-2,1}\left(\frac{6}{8}\right) & \psi^{\mathrm{PER}}_{-2,2}\left(\frac{6}{8}\right) & \psi^{\mathrm{PER}}_{-2,3}\left(\frac{6}{8}\right) \\ 1 & \psi^{\mathrm{PER}}_{0,0}\left(\frac{7}{8}\right) & \psi^{\mathrm{PER}}_{-1,0}\left(\frac{7}{8}\right) & \psi^{\mathrm{PER}}_{-1,1}\left(\frac{7}{8}\right) & \psi^{\mathrm{PER}}_{-2,0}\left(\frac{7}{8}\right) & \psi^{\mathrm{PER}}_{-2,1}\left(\frac{7}{8}\right) & \psi^{\mathrm{PER}}_{-2,2}\left(\frac{7}{8}\right) & \psi^{\mathrm{PER}}_{-2,3}\left(\frac{7}{8}\right) \end{pmatrix} = \frac{1}{\sqrt{8}} \begin{pmatrix} 1 & 1 & \sqrt{2} & 0 & 2 & 0 & 0 & 0 \\ 1 & 1 & \sqrt{2} & 0 & -2 & 0 & 0 & 0 \\ 1 & 1 & -\sqrt{2} & 0 & 0 & 2 & 0 & 0 \\ 1 & 1 & -\sqrt{2} & 0 & 0 & -2 & 0 & 0 \\ 1 & -1 & 0 & \sqrt{2} & 0 & 0 & 2 & 0 \\ 1 & -1 & 0 & \sqrt{2} & 0 & 0 & -2 & 0 \\ 1 & -1 & 0 & -\sqrt{2} & 0 & 0 & 0 & 2 \\ 1 & -1 & 0 & -\sqrt{2} & 0 & 0 & 0 & -2 \end{pmatrix}.$$

It is defined by

$$\psi_{-j,n}(i/N) = \frac{1}{\sqrt{N}} \begin{cases} 2^{-j/2} & \text{for } 0 \le t < s, \\ -2^{-j/2} & \text{for } s \le t < 2s, \\ 0 & \text{if } t < 0. \end{cases}$$

for $j = 0, 1, \ldots, J$, $n = 0, 1, \ldots, 2^j - 1$, $i = 0, 1, \ldots, N-1$, $s = 2^{J-j}$ and $t = i - 2sn$.

When the matrix Ψ generates Ψ_R: The trick for Haar

Remark

The Haar matrices Ψ and Ψ_R are *orthogonal*. That is,

$$\Psi\Psi^\top = \mathrm{Id} \quad \text{and} \quad \Psi_R\Psi_R^\top = \mathrm{Id}\,.$$

Crucial Lemma

Set $r = \lfloor \omega N \rfloor$ and let $P = (p_{i,j})$ be the permutation matrix such that $p_{i,j} = 1$ if and only if $j = i + r \pmod N$. Then,

$$\boxed{\Psi_R = P\Psi} \quad \Rightarrow \Psi\Psi_R^\top = P^\top.$$

Using Haar to solve the Invariance Equation: preconditioning

THE TRICK that makes everything easily computable

In the case of Haar we take

$$\mathbf{R}_{\text{prec}} = \Psi_R^\top \quad \text{which gives} \quad X_n = \Psi_R^\top Y_n$$

as a right preconditioner.

Then the system becomes

$$-\mathbf{F}_{\sigma,\varepsilon}(D_n^{\text{PER}}) = (\Psi_R - \Delta_{\sigma,\varepsilon}\Psi)\Psi_R^\top Y_n = (\text{Id} - \Delta_{\sigma,\varepsilon} P^\top) Y_n.$$

Using Haar to solve the Invariance Equation

To understand the effect of the above preconditioning let us look at the preconditioned matrix $\mathrm{Id} - \Delta_{\sigma,\varepsilon} P^\top$ for $N = 8$:

. . . and the matrix is. . .

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & f'_\sigma g_\varepsilon & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & f'_\sigma g_\varepsilon & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & f'_\sigma g_\varepsilon \\ f'_\sigma g_\varepsilon & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & f'_\sigma g_\varepsilon & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & f'_\sigma g_\varepsilon & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & f'_\sigma g_\varepsilon & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & f'_\sigma g_\varepsilon & 0 & 0 & 1 \end{pmatrix}$$

where the symbol $f'_\sigma g_\varepsilon$ in the row $i = 0, 1, \ldots, N-1$ denotes

$$f'_\sigma \left([\Psi \mathrm{D}^{\mathrm{PER}}]_i \right) g_\varepsilon(\theta_i).$$

As we see, this preconditioning (change of variables) is very good: by performing formally the Gauss Method on the system we obtain an explicit recurrence that solves the system in $\mathcal{O}(N)$ time.

A bootstrap on efficiency: the rotated wavelet coefficients

This preconditioner is so good that we should make the change of coordinates permanent and work with the *rotated wavelet coefficients*:

$$\mathrm{D}^{\mathrm{PER}}_{\mathrm{rot}} := \Psi_R \mathrm{D}^{\mathrm{PER}} \iff \mathrm{D}^{\mathrm{PER}} = \Psi_R^\top \mathrm{D}^{\mathrm{PER}}_{\mathrm{rot}} \iff \Psi \mathrm{D}^{\mathrm{PER}} = P^\top \mathrm{D}^{\mathrm{PER}}_{\mathrm{rot}}$$

This new approach has the following

SIMPLIFYING CONSEQUENCES

Reconstruction:

$$(\varphi(\theta_0), \varphi(\theta_1), \ldots, \varphi(\theta_{N-1})^\top = \Psi \mathrm{D}^{\mathrm{PER}}_{\mathrm{sol}} = \Psi \Psi_R^\top \mathrm{D}^{\mathrm{PER}}_{\mathrm{rot,sol}} = P^\top \mathrm{D}^{\mathrm{PER}}_{\mathrm{rot,sol}}.$$

Evaluation of $\mathbf{F}_{\sigma,\varepsilon}(\mathbf{D}^{\mathbf{PER}}_{\mathbf{n}})$ (the Invariance Equation):

$$\left[\mathbf{F}_{\sigma,\varepsilon}(\mathrm{D}^{\mathrm{PER}}_{\mathrm{n}})\right]_i = \left[\Psi_R \mathrm{D}^{\mathrm{PER}}_{\mathrm{n}}\right]_i - f_\sigma\left(\left[\Psi \mathrm{D}^{\mathrm{PER}}_{\mathrm{n}}\right]_i\right) \cdot g(\theta_i)$$

in rotated wavelet coefficients is equivalent to

$$\left[\mathrm{D}^{\mathrm{PER}}_{\mathrm{rot,n}}\right]_i - f\left(\left[P^\top \mathrm{D}^{\mathrm{PER}}_{\mathrm{rot,n}}\right]_i\right) \cdot g(\theta_i)$$

The rotated seed $\mathbf{D}^{\mathbf{PER}}_{\mathbf{rot,0}} := \Psi_R \mathbf{D}^{\mathbf{PER}}_{\mathbf{0}}$: Since

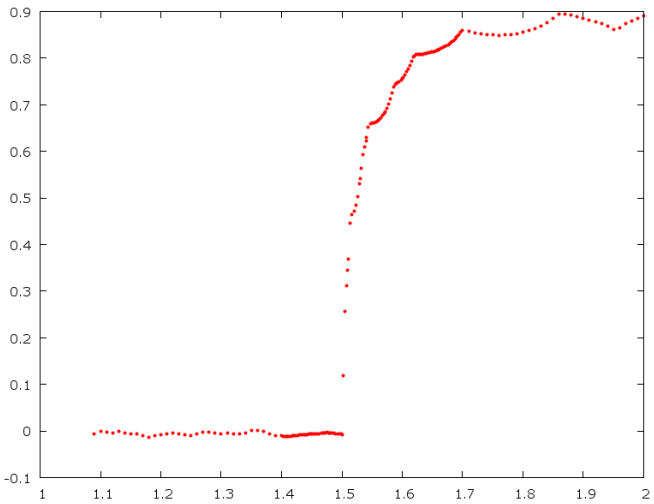
$$\Psi_R \Psi^\top = (\Psi \Psi_R^\top)^\top = (P^\top)^\top = P,$$

$$\mathrm{D}^{\mathrm{PER}}_{\mathrm{rot,0}} := \Psi_R \mathrm{D}^{\mathrm{PER}}_0 = \frac{1}{N} \Psi_R \Psi^\top (\varphi(\theta_0), \varphi(\theta_1), \ldots, \varphi(\theta_{N-1})^\top$$

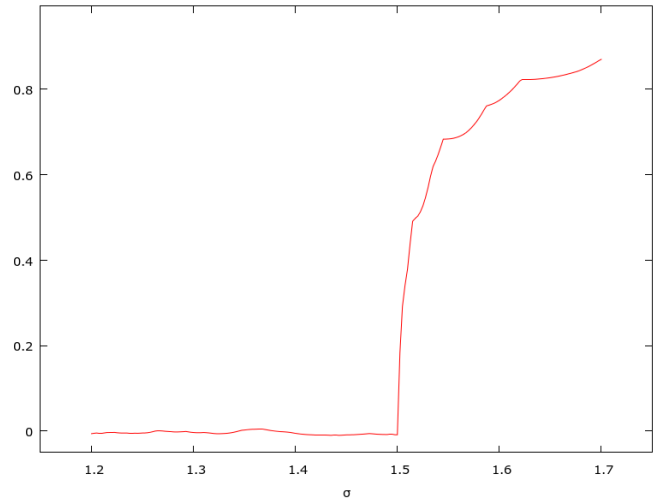
$$= \frac{1}{N} P (\varphi(\theta_0), \varphi(\theta_1), \ldots, \varphi(\theta_{N-1})^\top.$$

Using Haar to compute wavelet coefficients

Despite of the huge $(N \times N)$ linear systems to solve, as for the FWT, we can detect the pinching point in "$\mathcal{O}(N)$ time". Indeed, for $N = 2^{26}$ each Newton iterate takes less than 10 secs.



Regularity along $\varepsilon(\sigma)$.



Zoom around 1.5 the pinched point.

Because Haar it is not a basis of $\mathscr{B}^s_{\infty,\infty}$ (for $s > 0$), we need other Daubechies wavelets.

Using Daubechies to solve the Invariance Equation

The wavelets and rotated wavelets matrices Ψ_R and Ψ

Recall that we have to solve many times the system

$$(\Psi_R - \Delta_{\sigma,\varepsilon}\Psi)\mathsf{X}_\mathsf{n} = -\mathbf{F}_{\sigma,\varepsilon}(\mathsf{D}_\mathsf{n}^{\mathrm{PER}}),$$

now with Daubechies wavelets. To do so we need the *wavelets and rotated wavelets matrices* Ψ_R and Ψ defined by

$$\psi_{j,n}^{\mathrm{PER}}(\theta_i) = 2^{-j/2} \sum_{m\in\mathbb{Z}} \psi\left(\frac{(\theta_i + m) - 2^j n}{2^j}\right), \text{ and}$$

$$\psi_{j,n}^{\mathrm{PER}}(R_\omega(\theta_i)) = 2^{-j/2} \sum_{m\in\mathbb{Z}} \psi\left(\frac{(R_\omega(\theta_i) + m) - 2^j n}{2^j}\right),$$

for each $\theta_i = \frac{i}{N}$, $j = 0, \ldots, J$ and $n = 0, 1, \ldots, 2^j - 1$.

We have to take $N = 2^{J+1}$ to have the same number of equations and unknowns. Then the matrices are *huge of size* $2^{J+1} \times 2^{J+1}$ already for moderate values of J.

Using Daubechies to solve the Invariance Equation

The wavelets and rotated wavelets matrices Ψ_R and Ψ

The computation of the matrices Ψ_R and Ψ *is a massive computation which is extremely costly* in two ways:

- We have to evaluate a Daubechies wavelet on a point $2N^2$ times and, since Daubechies wavelets do not have a closed formula, the algorithm to perform these computations is an extreme CPU consuming process (fortunately the matrices Ψ and Ψ_R only need to be (pre)computed once).
- The storage of the matrices requires $2N^2$ memory slots. When, for instance, $N = 2^{26}$ and we store the wavelet values as `double` this would require
$$2 \cdot \left(2^{26}\right)^2 \cdot 2^3 \text{ bytes} = 2^{56} \text{ bytes} = 64 \text{ petabytes}$$
(that is, 64 mega gigabytes).

Fortunately there are a number of drastic simplifications that convert the problem into a feasible one.

Using Daubechies to solve the Invariance Equation

The wavelets and rotated wavelets matrices Ψ_R and Ψ

- To compute the matrices Ψ_R and Ψ of Daubechies wavelets with $p > 1$ vanishing moments *with high precision* we have adapted the $\mathbb{R}$-Daubechies–Lagarias algorithm to $\mathbb{S}^1$.
- To simplify the computations and the storage we have used the self-similarity of the matrices Ψ_R and Ψ.

[Daub] Daubechies, Ingrid, *Ten lectures on wavelets* Society for Industrial and Applied Mathematics (SIAM),Philadelphia, 1992, xx+357.

[Vid] Vidakovic, Brani, *Statistical modelling by wavelets* John Wiley & Sons, Inc., New York,1999, xiv+382.

Daubechies – Lagarias on the circle (on practice)

The matrix below illustrates the self-similarity of the matrices Ψ_R and Ψ (to save space we denote $\psi^{\mathrm{PER}}_{-j,n}$ by $\psi^{\mathrm{PER}}_{j,n}$).

Note that some of its vertical blocks are *sparse*:

The matrix is sparse for $j \geq 5$

1	$\psi^{\mathrm{PER}}_{0,0}((0)$	$\psi^{\mathrm{PER}}_{1,0}((0)$	$\psi^{\mathrm{PER}}_{1,1}((0)$	$\psi^{\mathrm{PER}}_{2,0}((0)$	$\psi^{\mathrm{PER}}_{2,1}((0)$	$\psi^{\mathrm{PER}}_{2,2}((0)$	$\psi^{\mathrm{PER}}_{2,3}((0)$	$\psi^{\mathrm{PER}}_{3,0}((0)$	0	0	0	0	0	0	$\psi^{\mathrm{PER}}_{3,7}((0)$
1	$\psi^{\mathrm{PER}}_{0,0}(\frac{1}{16})$	$\psi^{\mathrm{PER}}_{1,0}(\frac{1}{16})$	$\psi^{\mathrm{PER}}_{1,1}(\frac{1}{16})$	$\psi^{\mathrm{PER}}_{2,0}(\frac{1}{16})$	$\psi^{\mathrm{PER}}_{2,1}(\frac{1}{16})$	$\psi^{\mathrm{PER}}_{2,2}(\frac{1}{16})$	$\psi^{\mathrm{PER}}_{2,3}(\frac{1}{16})$	$\psi^{\mathrm{PER}}_{3,0}(\frac{1}{16})$	0	0	0	0	0	0	0
1	$\psi^{\mathrm{PER}}_{0,0}(\frac{2}{16})$	$\psi^{\mathrm{PER}}_{1,0}(\frac{2}{16})$	$\psi^{\mathrm{PER}}_{1,1}(\frac{2}{16})$	$\psi^{\mathrm{PER}}_{2,0}(\frac{2}{16})$	$\psi^{\mathrm{PER}}_{2,1}(\frac{2}{16})$	$\psi^{\mathrm{PER}}_{2,2}(\frac{2}{16})$	$\psi^{\mathrm{PER}}_{2,3}(\frac{2}{16})$	$\psi^{\mathrm{PER}}_{3,7}((0)$	$\psi^{\mathrm{PER}}_{3,0}((0)$	0	0	0	0	0	0
1	$\psi^{\mathrm{PER}}_{0,0}(\frac{3}{16})$	$\psi^{\mathrm{PER}}_{1,0}(\frac{3}{16})$	$\psi^{\mathrm{PER}}_{1,1}(\frac{3}{16})$	$\psi^{\mathrm{PER}}_{2,0}(\frac{3}{16})$	$\psi^{\mathrm{PER}}_{2,1}(\frac{3}{16})$	$\psi^{\mathrm{PER}}_{2,2}(\frac{3}{16})$	$\psi^{\mathrm{PER}}_{2,3}(\frac{3}{16})$	0	$\psi^{\mathrm{PER}}_{3,0}(\frac{1}{16})$	0	0	0	0	0	0
1	$\psi^{\mathrm{PER}}_{0,0}(\frac{4}{16})$	$\psi^{\mathrm{PER}}_{1,0}(\frac{4}{16})$	$\psi^{\mathrm{PER}}_{1,1}(\frac{4}{16})$	$\psi^{\mathrm{PER}}_{2,3}((0)$	$\psi^{\mathrm{PER}}_{2,0}((0)$	$\psi^{\mathrm{PER}}_{2,1}((0)$	$\psi^{\mathrm{PER}}_{2,2}((0)$	0	$\psi^{\mathrm{PER}}_{3,7}((0)$	$\psi^{\mathrm{PER}}_{3,0}((0)$	0	0	0	0	0
1	$\psi^{\mathrm{PER}}_{0,0}(\frac{5}{16})$	$\psi^{\mathrm{PER}}_{1,0}(\frac{5}{16})$	$\psi^{\mathrm{PER}}_{1,1}(\frac{5}{16})$	$\psi^{\mathrm{PER}}_{2,3}(\frac{1}{16})$	$\psi^{\mathrm{PER}}_{2,0}(\frac{1}{16})$	$\psi^{\mathrm{PER}}_{2,1}((0)$	$\psi^{\mathrm{PER}}_{2,2}(\frac{1}{16})$	0	0	$\psi^{\mathrm{PER}}_{3,0}(\frac{1}{16})$	0	0	0	0	0
1	$\psi^{\mathrm{PER}}_{0,0}(\frac{6}{16})$	$\psi^{\mathrm{PER}}_{1,0}(\frac{6}{16})$	$\psi^{\mathrm{PER}}_{1,1}(\frac{6}{16})$	$\psi^{\mathrm{PER}}_{2,3}(\frac{2}{16})$	$\psi^{\mathrm{PER}}_{2,0}(\frac{2}{16})$	$\psi^{\mathrm{PER}}_{2,1}(\frac{2}{16})$	$\psi^{\mathrm{PER}}_{2,2}(\frac{2}{16})$	0	0	$\psi^{\mathrm{PER}}_{3,7}((0)$	$\psi^{\mathrm{PER}}_{3,0}((0)$	0	0	0	0
1	$\psi^{\mathrm{PER}}_{0,0}(\frac{7}{16})$	$\psi^{\mathrm{PER}}_{1,0}(\frac{7}{16})$	$\psi^{\mathrm{PER}}_{1,1}(\frac{7}{16})$	$\psi^{\mathrm{PER}}_{2,3}(\frac{3}{16})$	$\psi^{\mathrm{PER}}_{2,0}(\frac{3}{16})$	$\psi^{\mathrm{PER}}_{2,1}(\frac{3}{16})$	$\psi^{\mathrm{PER}}_{2,2}(\frac{3}{16})$	0	0	0	$\psi^{\mathrm{PER}}_{3,0}(\frac{1}{16})$	0	0	0	0
1	$-\psi^{\mathrm{PER}}_{0,0}((0)$	$\psi^{\mathrm{PER}}_{1,1}((0)$	$\psi^{\mathrm{PER}}_{1,0}((0)$	$\psi^{\mathrm{PER}}_{2,2}((0)$	$\psi^{\mathrm{PER}}_{2,3}((0)$	$\psi^{\mathrm{PER}}_{2,0}((0)$	$\psi^{\mathrm{PER}}_{2,1}((0)$	0	0	0	$\psi^{\mathrm{PER}}_{3,7}((0)$	$\psi^{\mathrm{PER}}_{3,0}((0)$	0	0	0
1	$-\psi^{\mathrm{PER}}_{0,0}(\frac{1}{16})$	$\psi^{\mathrm{PER}}_{1,1}(\frac{1}{16})$	$\psi^{\mathrm{PER}}_{1,0}(\frac{1}{16})$	$\psi^{\mathrm{PER}}_{2,2}(\frac{1}{16})$	$\psi^{\mathrm{PER}}_{2,3}(\frac{1}{16})$	$\psi^{\mathrm{PER}}_{2,0}(\frac{1}{16})$	$\psi^{\mathrm{PER}}_{2,1}(\frac{1}{16})$	0	0	0	0	$\psi^{\mathrm{PER}}_{3,0}(\frac{1}{16})$	0	0	0
1	$-\psi^{\mathrm{PER}}_{0,0}(\frac{2}{16})$	$\psi^{\mathrm{PER}}_{1,1}(\frac{2}{16})$	$\psi^{\mathrm{PER}}_{1,0}(\frac{2}{16})$	$\psi^{\mathrm{PER}}_{2,2}(\frac{2}{16})$	$\psi^{\mathrm{PER}}_{2,3}(\frac{2}{16})$	$\psi^{\mathrm{PER}}_{2,0}(\frac{2}{16})$	$\psi^{\mathrm{PER}}_{2,1}(\frac{2}{16})$	0	0	0	0	$\psi^{\mathrm{PER}}_{3,7}((0)$	$\psi^{\mathrm{PER}}_{3,0}((0)$	0	0
1	$-\psi^{\mathrm{PER}}_{0,0}(\frac{3}{16})$	$\psi^{\mathrm{PER}}_{1,1}(\frac{3}{16})$	$\psi^{\mathrm{PER}}_{1,0}(\frac{3}{16})$	$\psi^{\mathrm{PER}}_{2,2}(\frac{3}{16})$	$\psi^{\mathrm{PER}}_{2,3}(\frac{3}{16})$	$\psi^{\mathrm{PER}}_{2,0}(\frac{3}{16})$	$\psi^{\mathrm{PER}}_{2,1}(\frac{3}{16})$	0	0	0	0	0	$\psi^{\mathrm{PER}}_{3,0}(\frac{1}{16})$	0	0
1	$-\psi^{\mathrm{PER}}_{0,0}(\frac{4}{16})$	$\psi^{\mathrm{PER}}_{1,1}(\frac{4}{16})$	$\psi^{\mathrm{PER}}_{1,0}(\frac{4}{16})$	$\psi^{\mathrm{PER}}_{2,1}((0)$	$\psi^{\mathrm{PER}}_{2,2}((0)$	$\psi^{\mathrm{PER}}_{2,3}((0)$	$\psi^{\mathrm{PER}}_{2,0}((0)$	0	0	0	0	0	$\psi^{\mathrm{PER}}_{3,7}((0)$	$\psi^{\mathrm{PER}}_{3,0}((0)$	0
1	$-\psi^{\mathrm{PER}}_{0,0}(\frac{5}{16})$	$\psi^{\mathrm{PER}}_{1,1}(\frac{5}{16})$	$\psi^{\mathrm{PER}}_{1,0}(\frac{5}{16})$	$\psi^{\mathrm{PER}}_{2,1}(\frac{1}{16})$	$\psi^{\mathrm{PER}}_{2,2}(\frac{1}{16})$	$\psi^{\mathrm{PER}}_{2,3}(\frac{1}{16})$	$\psi^{\mathrm{PER}}_{2,0}(\frac{1}{16})$	0	0	0	0	0	0	$\psi^{\mathrm{PER}}_{3,0}(\frac{1}{16})$	0
1	$-\psi^{\mathrm{PER}}_{0,0}(\frac{6}{16})$	$\psi^{\mathrm{PER}}_{1,1}(\frac{6}{16})$	$\psi^{\mathrm{PER}}_{1,0}(\frac{6}{16})$	$\psi^{\mathrm{PER}}_{2,1}(\frac{2}{16})$	$\psi^{\mathrm{PER}}_{2,2}(\frac{2}{16})$	$\psi^{\mathrm{PER}}_{2,3}(\frac{2}{16})$	$\psi^{\mathrm{PER}}_{2,0}(\frac{2}{16})$	0	0	0	0	0	0	$\psi^{\mathrm{PER}}_{3,7}((0)$	$\psi^{\mathrm{PER}}_{3,0}((0)$
1	$-\psi^{\mathrm{PER}}_{0,0}(\frac{7}{16})$	$\psi^{\mathrm{PER}}_{1,1}(\frac{7}{16})$	$\psi^{\mathrm{PER}}_{1,0}(\frac{7}{16})$	$\psi^{\mathrm{PER}}_{2,1}(\frac{3}{16})$	$\psi^{\mathrm{PER}}_{2,2}(\frac{3}{16})$	$\psi^{\mathrm{PER}}_{2,3}(\frac{3}{16})$	$\psi^{\mathrm{PER}}_{2,0}(\frac{3}{16})$	0	0	0	0	0	0	0	$\psi^{\mathrm{PER}}_{3,0}(\frac{1}{16})$

Using Daubechies to compute wavelet coefficients

- For Daubechies wavelets the right preconditioning does not work: We have to use $\Psi_R^\top$ as a *left preconditioner*.
  We have:

$$\Psi_R^\top(\Psi_R - \Delta_{\sigma,\varepsilon}\Psi)\mathsf{X}_\mathsf{n} = -\mathbf{F}_{\sigma,\varepsilon}(\mathsf{D}_\mathsf{n}^{\mathsf{PER}}).$$

Problem: $\Psi_R^\top\Psi_R$ is not the identity due to rounding errors because $\frac{1}{N}$ is of order the machine precision. The lower right part of the matrix $\Psi_R^\top\Psi_R$ has "garbage".

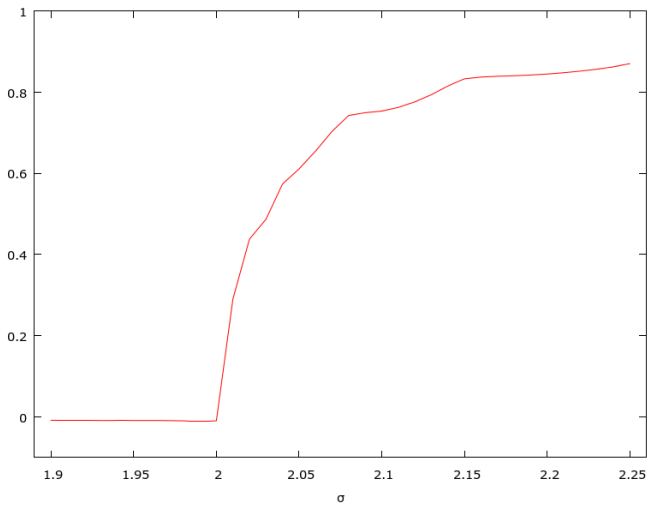
But anyway the system

$$\Psi_R^\top(\Psi_R - \Delta_{\sigma,\varepsilon}\Psi)\mathsf{X}_\mathsf{n} = -\mathbf{F}_{\sigma,\varepsilon}(\mathsf{D}_\mathsf{n}^{\mathsf{PER}}),$$

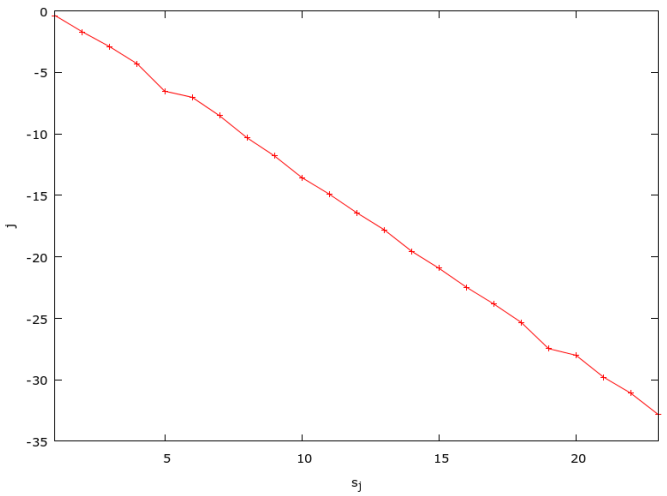
is easier to be solved with the use of *Krylov methods*: (more precisely with the TFQMR method with Arnoldi base construction).

Using Daubechies to compute wavelet coefficients

With these tools we get the following regularity graph of the Keller-GOPY attractor. The results are obtained by using a sample of 2^{24} points in $\mathbb{S}^1$ and the Daubechies Wavelet with 10 vanishing moments.



The detection of the regularity leap for another parameterization.



How we compute the regularity of a particular instance of φ.

As before, we can detect the pinched point in "in $\mathcal{O}(N)$ time" and with less iterates than Haar.

Conclusions

Our aim was the study of the use of wavelets in the numerical computation of invariant objects framework. That is, give a *generic way* to get $\varphi \sim d_0 + \sum_{\ell=0}^{N-1} d_\ell \psi_\ell^{\mathrm{PER}}(\theta)$. For us, φ is a SNA.

Theoretical point of view

1. Due to the geometry and topology of φ, we have introduced and justified the use of $\mathscr{B}^s_{\infty,\infty}$ in the SNA's framework.
2. Under "Keller's assumptions", we have *classified* $\varphi \in \mathscr{B}^s_{\infty,\infty}$ and related the wavelet coefficients of φ, $\mathrm{D}^{\mathrm{PER}}$, with such classification. Moreover, such relationship can be used, for example, when facing the fractalization route.
3. Due to the volume of calculations involved, we have introduced and justified the use of Newton's Method, Krylov methods and the FWT to calculate $\mathrm{D}^{\mathrm{PER}}$ in our framework.

Conclusions

Theoretical point of view

5. With respect to Newton's Method, we have related the use of the Trapezoidal rule with the initial seed $D_{0)}^{PER}$.
6. Moreover, in the Haar's case we have related λ_φ with the convergence of Newton's Method and, also, we have found an explicit solution of the linear system, via a permutation matrix P (and a precondition strategy).
7. We have proved that we can take the orbit of a point as $a\_{J}[n]$ (the *initial seed* of the FWT).

Conclusions

Algorithmic point of view

1. We have expressed the Invariance Equation as "matrix×vector". Using the same idea (and the same goals), we have *compacted* the Jacobian matrix $\mathbf{JF}_{\sigma,\varepsilon} = \Psi_R - \Delta_{\sigma,\varepsilon}\Psi$.
2. To work and compute with Ψ and Ψ_R in the Daubechies case, we have *rephrased* the Daubechies – Lagarias algorithm from $\mathbb{R}$ to $\mathbb{S}^1$. Using it and the inherited properties of the Daubechies wavelets, we have derived properties of Ψ and Ψ_R.
3. Moreover, we have found *good* precondition strategies to solve the system in a feasible way. As a consequence, we can go fast and deep. In particular, when $\psi(x)$ is the Haar wavelet, we have performed a strategy to get *the* explicit solution.
4. Focusing in the FWT performance, we have sorted a big signal of the attractor φ faster than "fast sorting algorithms" using Birkhoff's Ergodic Theorem.